Linear Algebra

Rose-Hulman Institute of Technology
R.J. Marks II Class Notes
(1970)

NEAR ALGEBRA

12-9-70

$$\begin{array}{cc} & \leftarrow \text{ROW} \rightarrow \\ \updownarrow \text{COLUMB} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & \underset{(4,2)}{a_{42}} \end{bmatrix} \end{array} \Rightarrow 4 \times 3 \text{ MATRIX}$$

(ROWS) x (COLUMBS)

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$C = A + B = \begin{bmatrix} b_{11} + a_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ b_{21} + a_{21} & a_{22} + b_{22} & a_{23} + b_{23} \end{bmatrix}$$

MATRICES COMMUTATIVE, ie $A + B = B + A$

$$A = \begin{bmatrix} a_{1} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$A + B$ IS NOT DEFINED

$(A+B)+C = A+(B+C)$ $\forall$ A, B, C $\ni$ ALL ARE $n \times m$ MATRICES

A ZERO MATRIX HAS AS ALL ITS ELEMENTS IS ZERO

NEGATIVE MATRIX - REVERSE SIGNS OF ALL ELEMENTS

MULTIPICATION OF MATRIX $\$ \#$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow 2A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow 2A = \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$$

$$\therefore \quad pA = \begin{bmatrix} pa & pb \\ pc & pd \end{bmatrix}$$

ALSO $p(qA) = (pq)A$

$pA + pB = p(B+A)$

$pA + qA = A(p+q)$

Pg 21 1,2

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HAVE DEFINED ADDITION & MULTIPLICATION

$C = A + B \Rightarrow a_{ij} + b_{ij} = c_{ij}$

$D = A - B \Rightarrow d_{ij} = a_{ij} - b_{ij}$

DIVISION

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

WISH TO FIND $X \ni AX = I$ $\quad I = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad \begin{matrix} \Rightarrow BI = B \\ IB = B \end{matrix}$

<u>CONSIDERING SQUARE MATRIX (nxn)</u>

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} X & Y \\ Z & U \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$3X + 4Z = 1;\ 2X + 3Z = 0 \Rightarrow X = 3;\ Z = -2$

$3Y + 4U = 0;\ 2Y + 3U = 1 \Rightarrow Y = -4;\ U = 3$

$$\therefore X = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

CHECK:

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

NOTATION: $AA^{-1} = I$

TRY $A^{-1}A$

$$\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow AA^{-1} = A^{-1}A = I$$

FOR SQUARE MATRICES

GIVEN $AA^{-1} = I$ & $A^*A = I$, SHOW $A^* = A^{-1}$

$$\left.\begin{matrix} A^*AA^{-1} = (A^*A)A^{-1} = IA^{-1} = A^{-1} \\ A^*AA^{-1} = A^*(AA^{-1}) = A^*I = A^* \end{matrix}\right\} \text{FOR SQUARE MATRICES}$$

DOES EVERY MATRIX HAVE AN INVERSE?

$$\text{LET} \quad \overset{A}{\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}}\begin{bmatrix} X & Y \\ Z & U \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow 4X + 2Z = 1;\ 2X + Z = 0$

WHICH CANNOT BE SOLVED

MATRIX HAS INVERSE IF DETERMINANT IS NOT ZERO

INVERSE $\Rightarrow$ NON-SINGULAR

NO INVERSE $\Rightarrow$ SINGULAR

NOTATION: $\frac{A}{B} \overset{?}{=} AB^{-1}$; $\frac{A}{B} \overset{?}{=} B^{-1}A$

$\therefore$ DROP $\frac{A}{B}$ NOTATION

EXAMPLE: $3X + 4Y = 2$; $2X + 3Y = 1$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Leftrightarrow AU = p$$

$$A^{-1}AU = A^{-1}p \Rightarrow U = A^{-1}p$$

FROM PRECEEDING PAGE: $A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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$$AA^{-1} = I \Rightarrow A^{-2} = A^{-1}A^{-1}$$

$$(A+B)^2 \neq A^2 + 2AB + B^2$$

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= (A+B)A + (A+B)B \\ &= A^2 + BA + AB + B^2 \end{aligned}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$J^2 = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$aI + bJ = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$(aI + bJ)(cI + dJ) = (ac - bd)I + (ad + bc)J$$

$\Rightarrow aI$ REPRESENTS REAL #'s

bJ REPRESENTS IMAGINARY #'s

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$\Rightarrow \cos\theta\, I + \sin\theta\, J \Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = e^{J\theta}$$

EX)
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + x_3 &= -1 \\ x_1 + x_2 \quad &= 1 \end{aligned} \Rightarrow \underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{p}{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

$$\Rightarrow Ax = p \Rightarrow A^{-1}Ax = A^{-1}p \Rightarrow x = A^{-1}p$$

SOLVING DIRECTLY FROM EQUATIONS: $x_1 = 1;\ x_2 = 0;\ x_3 = -1$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

EX)
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_2 + x_3 &= -1 \\ x_1 + x_2 \quad &= 1 \end{aligned}$$

LINEAR DEPENDENT EQUATIONS

LET $x_2 = \alpha \quad x_1 = 1 - \alpha \quad x_3 = -1 - \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-\alpha \\ \alpha \\ -1-\alpha \end{bmatrix}$$

$x = A^{-1}p \Rightarrow x$ IS UNIQUE

HOWEVER, x IS NOT UNIQUE $\Rightarrow A^{-1}$ NOT UNIQUE $\Rightarrow A^{-1}$ DOESN'T EXIST

EX)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ x_2 + x_3 &= -1 \\ x_1 + x_2 \quad &= 1 \\ x_1 + 2x_2 + x_3 &= 0 \end{aligned}$$

CANNOT EXIST $\Rightarrow$ INVERSE OF MATRIX EQUIVALENT IS NOT UNIQUE

(RIGHT ON!)

SIMILAR TO FINDING INTERSECTION OF 3 PLANES
(X, Y, Z) FOR 3X3 MATRIX

12-16-70

Pg 38 #4

$$Ax=b$$

ASSUME X HAS TWO SOLUTIONS: U & V

$\exists\ Au=b;\ Av=b \Rightarrow Au-Av=0 \Rightarrow A(u-v)=0$

$Au=b;\ A[\alpha(u-v)]=b \Rightarrow A[u-\alpha(u-v)]=b$

$\therefore$ HAS OVER TWO SOLUTIONS

X, THEN HAS 0, 1 OR AN INFINITE # OF SOLUTIONS

A) $\begin{cases} x_1+x_2+5x_3=11 \\ 2x_1+x_2+7x_3=15 \\ 2x_1 \qquad +4x_3=8 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 2 & 1 & 7 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 8 \end{bmatrix}$

B) $\begin{cases} x_1+x_2+5x_3=11 \\ x_2+3x_3=7 & E_2 \to 2E_1-E_2 \\ 2x_2+6x_3=14 & E_3 \to 2E_1-E_2 \end{cases}$

C) $\begin{cases} x_1+x_2+5x_3=11 \\ x_2+3x_3=7 & E_2 \to E_3-2E_2 \\ 0=0 \end{cases}$

D) $\begin{cases} x_1+ \qquad 2x_3=4 & E_1 \to E_1-E_2 \\ x_2+3x_3=7 \\ 0 \qquad =0 \end{cases}$

ALL OF THESE EQUATIONS HAVE SAME SOLUTIONS

$x_1=4-2x_3\ ;\ x_2=7-3x_3$

MATRIX EQUIVALENT

A) $\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 5 & 1 \\ 2 & 1 & 7 & 5 \\ 2 & 0 & 4 & 8 \end{array}\right]}_{\text{AUGMENTED MATRIX}} \xrightarrow[R_3 \to 2R_1-R_2]{R_3 \to 2R_1-R_3}$ B) $\left[\begin{array}{ccc|c} 1 & 1 & 5 & 11 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 6 & 14 \end{array}\right]$

C) $\xrightarrow{R_3 \to R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 5 & 11 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

WANT AS MANY 0'S IN LWR LFT CORNER

OR GENERAL FORM $\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & \cdots & x_n & \\ 1 & - & \cdots & & - & b_1 \\ 0 & 1 & \cdots & & & b_2 \\ 0 & 0 & 1 & \cdots & & b_3 \\ \vdots & \vdots & \vdots & 1 & a_{n-1,n} & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_n \end{array}\right] \Rightarrow x_n = b_n$

$x_{n-1} + a_{n-1,n} x_n = b_{n-1}$

$\therefore$ MAY "WORK WAY UP" TO b_1

DON'T MANIPULATE COLUMNS $\Rightarrow$ WILL INTRODUCE NEW VARIABLES

DID NOT WORK ON $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$ BECAUSE EQUATIONS

$\neq 1$

DON'T HAVE UNIQUE SOLUTION $= \begin{bmatrix} 4-2\alpha \\ 7-3\alpha \\ \alpha \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

PROBLEMS: SOLVE EQ. OF EX. 3 ON PG. 38 IN ABOVE MANNER

(ON TEST)

11-17-70)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3 THINGS CAN DO WITH MATRIX

1) INTERCHANGE ROWS
2) MULTIPLY A ROW BY K
3) ADDING A MULTIPLE OF ONE ROW TO ANOTHER

I) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ IS TYPE 1 (E_{13}) OR E_I

II) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$

$\Uparrow$

IS TYPE 2, DENOTED $F_3(k)$

3) $\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+ka_{11} & a_{22}+ka_{12} & a_{23}+ka_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

↖ IS TYPE 3, DENONED $G_{21}(k)$

DEF - ANY TWO MATRICES THAT CAN BE OBTAINED FROM EACH OTHER ARE CALLED EQUIVELENT

E_{13}, $F_3(k)$, & $G_{21}(k)$ ARE CALLED EQUIVELENT MATRICES.

EQUIVALENCE ⇒ A ACTED ON BY A NUMBER OF ELEMENTRY PROPERTIES YIELDS B, THEN A BE THUS YIELDED FROM B

(EX) $F_4(3)\, G_{21}(3)\, E_{12}\, A = B$

$\Rightarrow A = E_{12}^{-1}\, G_{21}^{-1}(3)\, F_4^{-1}(3)\, B$

ASSUMING RECIPRICALS OF EQ. MATRICES EXIST.

A) $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad E_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow E_{ij}^{-1} = E_{ij}$$

B) $F_2(\frac{1}{k}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix} = F_2(\frac{1}{k})\, F_2(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow F_i^{-1}(k) = F_i(\tfrac{1}{k})$$

C) $G_{21}(k) = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = G_{21}(-k)\, G_{21}(k) = I$

$$\Rightarrow G_{ij}^{-1}(k) = G_{ij}(-k)$$

$A \sim B$ AND $B \sim C \Rightarrow A \sim C$

READ "IS EQUIVALENT TO"

12-21-70

$$AB = A[b_{*1}\ b_{*2} \ldots] = [Ab_{*1}\ Ab_{*2}\ Ab_{*3} \ldots]$$

Ex $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{12} & x & x \\ a_{21}b_{11}+a_{22}b_{22} & x & x \end{bmatrix}$$

$$A[B|b] = [AB|Ab]$$

ROW ECHELON FORM

1 b) Pg 53

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 5 & 7 & 9 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow[R_2 \to -R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

MAKE $a_{11} = 1$ LOWER LEFT CORNER = 0

MAKE $a_{*1} = 0$

MAKE $a_{22} = 1$ ECT.

MAY USE $a_{*2} = 1$ TO MAKE TERM ABOVE IT 0

ANY 1 CAN MAKE COLOMN, (SAVE 1) = 0

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

ASSIGNMENT: Pg 53 1 a, c, e, g 2a

ROW EQUIVALENCE $\rightarrow \exists$ COLUMN EQUIVALANCE

EX) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \overset{c}{\cong} \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \overset{c}{\cong} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

COLUMN EQUIVALENCE MAY BE USED ON MATRICES AS ROW EQUIVALENCE

Pg 48

$Ax = b \Rightarrow PAx = Pb$

1-5-71 JAN 1

COLONIAL .2 C_0

TOP TASTE .3 T_0

WONDER .5 W_0

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{21} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

a_{mm} IS THE FRACTION OF m'S CUSTOMERS THAT REMAIN

a_{mn} " " " " n'S " " SWITCH (TO m)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \overset{x_0}{\begin{bmatrix} C_0 \\ T_0 \\ W_0 \end{bmatrix}} \qquad \text{LET } A = \begin{bmatrix} .8 & .2 & .1 \\ .1 & .7 & .3 \\ .1 & .1 & .6 \end{bmatrix}$$

$$Ax_0 = x_1 = \begin{bmatrix} .27 \\ .38 \\ .35 \end{bmatrix} \qquad x_2 = Ax_1 = \begin{bmatrix} .327 \\ .398 \\ .275 \end{bmatrix} \qquad x_4 = A^4 x_0 = A^2 A^2 x_0$$

$$\Rightarrow x_4 = \begin{bmatrix} .397 \\ .384 \\ .219 \end{bmatrix} \qquad x_8 = A^4 A^4 x_0 = \begin{bmatrix} .442 \\ .367 \\ .201 \end{bmatrix} \qquad x_{16} = \begin{bmatrix} .450 \\ .350 \\ .200 \end{bmatrix}$$

$\therefore$ IT REACHES A LIMIT

$\lim_{n \to \infty} A^n x_0$ EXISTS

(CONT.) TWO BASIC ASSUMPTIONS

1) $c_0 + t_0 + w_0 = 1$; (ALSO $c_n + t_n + w_n = 1$)

2) Σ EACH COLUMN = 1 OR $a_{1n} + a_{2n} + a_{3n} = 1$

$$Ax_n = x_{n+1} \Rightarrow a_{11}c_n + a_{12}t_n + a_{13}w_n$$

$$a_{21}c_n + a_{22}t_n + a_{23}w_n$$

$$a_{31}c_n + a_{32}t_n + a_{33}w_n$$

$$\Rightarrow (a_{11} + a_{21} + a_{31})c_n + (a_{12} + a_{22} + a_{32})t_n + (a_{13} + a_{23} + a_{33})w_n$$

$$= c_n + t_n + w_n = c_{n+1} + t_{n+1} + w_{n+1} = 1$$

$\therefore$ THE PARENTHESISED PART OF PART 1 IS PROVED BY MATHEMATICAL INDUCTION

ASSUME: $0 \leq a_{ij} \leq 1$

THEM:

IF $a_{1n} + a_{2n} + a_{3n} + \ldots = 1$, AND $0 \leq a_{nm} \leq 1$

THEN $\lim_{n\to\infty} A^n$ EXISTS, CALL $\lim_{n\to\infty} A^n = \bar{A}$

LET $x_n = \bar{A}x_0 \Rightarrow Ax_n \cong x_{n+1} \cong x_n$

$\lim_{n\to\infty} x_n = x \Rightarrow x = Ax \Rightarrow (A - I)x = 0$

EX) $$\begin{bmatrix} .8 & .2 & .1 \\ .1 & .7 & .3 \\ .1 & .1 & .6 \end{bmatrix} \begin{bmatrix} c \\ t \\ w \end{bmatrix} = \begin{bmatrix} c \\ t \\ w \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} .8c + .2t + .1w &= c \\ .1c + .7t + .3w &= t \\ .1c + .1t + .6w &= w \end{aligned}\right\} \Rightarrow \begin{aligned} -.2c + .2t + .1w &= 0 \\ .1c - .3t + .3w &= 0 \\ .1c + .1t - .4w &= 0 \end{aligned}$$

THREE HOMOGENENEOUS EQUATIONS (CAN'T SOLVE)

THIRD EQUATION (TA, TA!) $c + t + w = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -2 & 2 & 1 & 0 \\ 1 & -3 & 3 & 0 \end{array}\right] \Rightarrow \begin{aligned} c &= .45 \\ t &= .35 \\ w &= .20 \end{aligned}$$

FOR THE GENERAL CASE:

$$\left.\begin{array}{l} c + t + w = 1 \\ (a_{11}-1)c + a_{12}t + a_{13}w = 0 \\ a_{21}c + (a_{22}-1)t + a_{23}w = 0 \\ a_{31}c \quad a_{32}t + (a_{33}-1)w = 0 \end{array}\right\} \text{STRIKE OUT OUT}$$

$$\bar{A} = \lim_{n\to\infty} A^n = \begin{bmatrix} c & c & c \\ t & t & t \\ w & w & w \end{bmatrix} \quad \text{WHEN} \quad \lim_{n\to\infty} x_n = x = \begin{bmatrix} c \\ t \\ w \end{bmatrix}$$

PROVE $\lim A^{n+1} = \lim A^n$ AND $\bar{A}A = \bar{A}$

1-6-70

1-8-70

$0 < a_{ij} \leq 1$ INSTEAD OF $0 < a_{ij} < 1$

CONSIDER ⇒ $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ETC.

Pg 68, #3

$$A = \begin{bmatrix} a_{1\bullet} \\ a_{2\bullet} \\ \vdots \\ 0 \\ \vdots \\ a_{n\bullet} \end{bmatrix} B = \begin{bmatrix} a_{1\bullet}B \\ a_{2\bullet}B \\ \vdots \\ 0 \\ \vdots \\ a_n B \end{bmatrix} \neq I$$

Pg 68, #5 $AB = C$

ASSUMTION: C IS REGULAR ⇒ A & B ARE REGULAR (i.e. NONSINGULAR)

C REG ⇒ $\exists\ C^{-1} \ni CC^{-1} = CC^{-1} = I$

PROOF $AB = C$

$$C^{-1}(AB) = C^{-1}C \Rightarrow (C^{-1}A)B = I \Rightarrow B^{-1} = C^{-1}A$$

SIMILARLY: $AB = C \Rightarrow (AB)C^{-1} = CC^{-1} = A(BC^{-1}) = I \Rightarrow A^{-1} = BC^{-1}$

1) $\{x+y=2;\ 2x+2y=3\}$

INCONSISTENT SYSTEM OF EQUATIONS

2) $\{x+y=2;\ x+2y=3\}$

ONE SOLUTION

3) $\{x+y=2;\ 2x+2y=4\}$

INFINITE SOLUTIONS

EX) $\left[\begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 2 & 5 & 0 & 1 \\ 5 & 13 & 7 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & \frac{1}{2} & \frac{1}{2} \\ 3 & 5 & 0 & 1 \\ 5 & 13 & 7 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 3 & \frac{9}{2} & \frac{3}{2} \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 3/2 & \frac{1}{2} \\ 0 & 3 & 9/2 & \frac{3}{2} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 2 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 3/2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & -\frac{1}{2} \\ 0 & 1 & 3/2 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{cases} x - \frac{5}{2}z = -\frac{1}{2} \\ y + \frac{3}{2}z = \frac{1}{2} \\ 0 + 0 = 0 \end{cases}$ LET $z = \lambda \Rightarrow \begin{cases} x = \frac{5}{2}\lambda - \frac{1}{2} \\ y = -\frac{3}{2}\lambda + \frac{1}{2} \end{cases}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{2}\lambda - \frac{1}{2} \\ -\frac{3}{2}\lambda + \frac{1}{2} \\ \lambda \end{bmatrix} = \begin{bmatrix} \frac{5}{2}\lambda \\ -\frac{3}{2}\lambda \\ \lambda \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -3/2 \\ 1 \end{bmatrix}\lambda + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$

$\therefore$ A ROW IN R.E.F. $\Rightarrow$ INFINITE SOLUTIOR

(ONE ROW IS A LINEAR COMBINATION OF THE OTHERS)

1-11-71

INCONSISTENT SET OF EQUATIONS:

EX) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -4 & -5 \\ 0 & 6 & 8 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & -4/3 & -5/3 \\ 0 & 0 & 0 & 11 \end{array}\right]$

$\Rightarrow 0x + 0y + 0z = 11$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 7/3 & 17/3 \\ 0 & 1 & -4/3 & -5/3 \\ 0 & 0 & 0 & 11 \end{array}\right]$

RANK OF A MATRIX

PROVISIONAL DEFINITION: THE ROW RANK OF A MATRIX IS THE NUMBER OF ROWS NOT CONSISTING OF ZEROS EXCLUSIVELY

A NONHOMOGENEOUS SYSTEM OF EQUATIONS IS CONSISTANT IF THE RANK OF THE MATRIX OF COEFFICIENTS EQUALS THE RANK OF THE AUGMENTED MATRIX

$\therefore$ ALL SQUARE MATRICES HAVE A REF OF I

$A \in F_{nn} \ni A^{-1}$ EXISTS; $r(A) = n \Rightarrow A \sim I_n$ (REF)

$AX = I$

$AX_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad AX_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ETC

INCONSISTANT:

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 2 & 3 & 1 & -1 & 0 \\ 3 & 4 & -1 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 1 & 5 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -7 & 4 & 0 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ INFINITE SOLUTIONS

$$\left.\begin{array}{l} X_1 - 7X_3 + 4X_4 = 0 \\ X_2 + 5X_3 - 3X_4 = 0 \end{array}\right\} \rightarrow \begin{array}{l} X_1 = 7X_3 - 4X_4 \\ X_2 = -5X_3 + 3X_4 \end{array}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 7X_3 - 4X_4 \\ -5X_3 + 3X_4 \\ X_3 \\ X_4 \end{bmatrix} = X_3 \begin{bmatrix} 7 \\ -5 \\ 1 \\ 0 \end{bmatrix} + X_4 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 7 \\ -5 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

RANK = # EQUATIONS - # OF UNKNOWNS

EX) $\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 4 \\ 2 & 3 & 1 & -1 & 10 \\ 3 & 4 & 1 & -1 & 14 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -7 & 4 & 0 \\ 0 & & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 - 7x_3 + 4x_4 = 2$

$x_2 + 5x_3 - 3x_4 = 2$

$\Rightarrow x_1 = 7x_3 - 4x_4 + 2$

$x_2 = -5x_3 + 3x_4 + 2$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 6 \end{bmatrix} + c_1 \begin{bmatrix} 7 \\ 5 \\ 0 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$x = x_p + x_c$$

1-13-70

1) ADDING MULTIPLE OF ONE ROW WITH ANOTHER, DOESN'T EFFECT DETERMINATE

2) SWITCHING 2 ROWS CHANGES SIGN

3) MULTIPLYING ROW OR COLUMN BY K, MULTIPLYS DETERMINATE VALUE BY K

TWO EQUAL ADJACENT COLUMNS ⇒ DETERMINATE VANISHES

$\underline{B} = [b_{\cdot 1} \dots b_{\cdot k}\, b_{\cdot k} \dots b_{\cdot n}] \Rightarrow \det B = 0$

$\left.\begin{array}{l} B = [b_{\cdot 1} \dots b_{\cdot k}\, b_{\cdot k+1} \dots b_{\cdot n}] \\ \underline{C} = [b_{\cdot 1} \dots b_{\cdot k+1}\, b_{\cdot k} \dots b_{\cdot n}] \end{array}\right\} \det B = -\det C$

$B = [b_{\cdot 1} \dots b_{\cdot n}]$ IF $\cdot i = \cdot j \Rightarrow \det B = 0$

SWAPPING COLUMNS: ANY 2 COLUMNS SWAPPED ⇒ DETERMINATE HAS SIGN CHANGE

$B = [b_{\cdot 1}\, b_{\cdot 2} \dots b_{\cdot k} \dots b_{\cdot n}]$

$C = [b_{\cdot 1}\, b_{\cdot 2} \dots c_{\cdot k} \dots c_{\cdot n}]$

$A = [b_{\cdot 1}\, b_{\cdot 2}\, \beta b_{\cdot k} + \gamma c_{\cdot k} \dots b_{\cdot n}]$

$\det A = \beta \det B + \gamma \det C$

$$A = [a_{\cdot 1}\ a_{\cdot 2} \ldots a_{\cdot k} + \gamma a_{\cdot \ell} \ldots a_{\cdot n}]$$

$$\det A = \det[a_{\cdot 1} \ldots a_{\cdot k} \ldots a_{\cdot n}] + \gamma \cancelto{0}{\det} [a_{\cdot 1} \ldots \underset{\uparrow \text{column } k}{a_{\cdot \ell}} \ldots a_{\cdot n}]$$

$$= \det [a_{\cdot 1} \ldots a_{\cdot k} \ldots a_{\cdot n}]$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{12} \overset{-}{\begin{bmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}} \cdots$$

MAY ALSO BE EXPANDED VIA COLUMNS

$$\det A = a_{11} \begin{vmatrix} a_{22} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots$$

$$= |A_{11}| - |A_{12}| + \ldots$$

$(-1)^{i+k}$ MINOR A_{ik} = COFACTOR $|A_{ik}|$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

PROVE $\det A$ IS SAME FOR EXPANDING DETERMINATE BY MATH. INDUCTION
(HE DOED IT!)

1-15-71

TEST MONDAY; PROOFS SIMPLE
SOLN. OF LIN. SYSTEM BY COLUMN VECTORS
DETERMINATES

LECTURE:

$\det AB = \det A \det B \ni A \in$ ELEMENTRY MATRICES

PROOF: $\det EB = \det E \det B$ $\qquad \det I = 1$

$\det FB = \det F \det B$ $\qquad \det E = -1$

$\det GB = \det G \det B$

$\det E \det B = -\det B$

$\det EB = -\det B$

SIMILAR FOR F & G

J IS nn MATRIX HAVING RANK<n

$$J = \begin{bmatrix} 1 & x & 0 & x & x & \cdots \\ 0 & 0 & 1 & \cdots & & \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ EXPANDING BY BOTTOM ROW

det J = 0

THEOREM, IF J IS A SQUARE MATRIX IN REF WITH RANK SMALLER THAN ITS ORDER, THEN det J = 0

PROOF:

$$J = \begin{bmatrix} x & & x & x & \cdots \\ & & & & \\ 0 & 0 & 0 & & \end{bmatrix} = \begin{bmatrix} J_{1\cdot} \\ J_{2\cdot} \\ 0 \end{bmatrix}$$

EXPAND BY BOTTOM ROW

THEOREM: det JB = 0

PROOF:

$$JB = \begin{bmatrix} J_{1\cdot} \\ J_{2\cdot} \\ \vdots \\ 0 \end{bmatrix} B = \begin{bmatrix} J_{1\cdot}B \\ J_{2\cdot}B \\ \vdots \\ 0 \end{bmatrix} = 0$$

EXPAND det JB w.r.t. ELEMENTS IN BOTTOM ROW

THEOREM: $A \overset{R}{\sim} J \rightarrow \det A = 0$

PROOF: $A = E_1 E_2 E_3 \cdots E_n J \ni E_i \in$ ELEMENTARY MATRICES

$$\begin{aligned} \det A &= \det(E_1 E_2 E_3 \ldots E_n) J \\ &= \det E_1 \det(E_2 \ldots E_n) J \\ &\vdots \\ &= \det E_1 \det E_2 \ldots \det E_n \overset{0}{\det J} = 0 \end{aligned}$$

THEOREM: $\det AB = \det A \det B$

PROOF: (1) $A \overset{R}{\sim} I \Rightarrow A = E_1 E_2 \ldots E_n \ni E_i (i = 1, 2 \ldots n) \in (EMS)$

$\Rightarrow \det AB = \det (E_1 E_2 \ldots E_n) B$

$= \det E_1 \det E_2 \ldots \det E_n \det B$

$= \det (E_1 E_2) \det (E_3 \ldots E_n) B$

$= \det (E_1 E_2 E_3 \ldots E_n) \det B$

$= \det A \det B$

(2) $A \overset{R}{\sim} J \rightarrow \det A = 0$

$\det AB = \det (E_1 E_2 \ldots E_n) J$

$= \det E_1 \det E_2 \ldots \det E_n \det \overset{\widetilde{A}^{0}}{JB} = 0$

$$\det A = a_{11} |A_{11}| + a_{12} |A_{12}| + a_{13} |A_{13}|$$
$$= a_{21} |A_{21}| + a_{22} |A_{22}| + a_{23} |A_{23}|$$

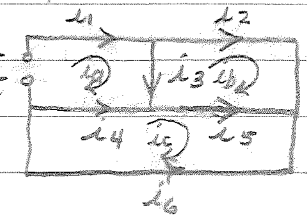
$$\left| \begin{array}{cccc} + & - & + & - \\ - & + & - & \\ + & - & + & \\ - & + & & \end{array} \right.$$ ← COFACTOR SIGN

$$\det A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{bmatrix} |A_{11}| \\ |A_{12}| \\ \vdots \\ |A_{1n}| \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{bmatrix} |A_{21}| \\ |A_{22}| \\ \vdots \\ |A_{2n}| \end{bmatrix}$$

$$\begin{vmatrix} |A_{11}| & \cdots & |A_{n1}| \\ |A_{12} & & \vdots \\ \vdots & & \vdots \\ |A_{1n}| & & |A_{nn}| \end{vmatrix} = \operatorname{adj} A$$ (ADJOINT OR ADJUGATE)

$$A \cdot \operatorname{adj} A = \det A \cdot I$$

1-26-71



BY KIRCHHOFF'S LAWS

$i_1 = i_a$ $\quad i_4 = i_c - i_a$

$i_2 = i_b$ $\quad i_5 = i_c - i_b$

$i_3 = i_a - i_b$ $\quad i_6 = -i_c$

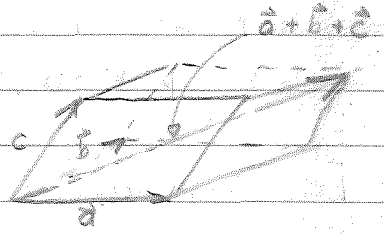
$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}$$

VECTORS

1) $\vec{a} + \vec{b}$ $\quad \vec{a} \in V$ & $\vec{b} \in V \Rightarrow \vec{a} + \vec{b} \in V$ & UNIQUE

2) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

3) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$



4) $\lambda \vec{a} \Rightarrow$ MAGNITUDE $\cdot \lambda$; SAME DIRECTION

5) $\lambda(\vec{a} + \vec{b}) = \lambda \vec{a} + \lambda \vec{b}$

6) $\lambda(\mu \vec{a}) = (\lambda \mu)\vec{a}$

7) $\exists\ \vec{0} \ni \vec{a} + \vec{0} = \vec{a}$

8) $\exists\ \vec{a}' \ni \vec{a}' + \vec{a} = \vec{0}$

9) $(\lambda + \mu)\vec{a} = \lambda \vec{a} + \mu \vec{a}$

DOT PRODUCT OF 2 VECTORS

1) $\vec{a} \cdot \vec{b} \in R \ni R \in$ REAL NUMBERS

2) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

3) $\vec{a} \cdot \vec{b} \cdot \vec{c}$ NOT DEFINED, EVEN WITH BRACKETS

(CONT.)

4) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

5) $\lambda(\vec{a}\cdot\vec{b}) = (\lambda\vec{a})\cdot\vec{b}$

6) $\vec{a}\cdot\vec{b}\,\vec{c} = (\vec{a}\cdot\vec{b})\vec{c}$ ANALOGOUS TO $\lambda\vec{c} = \vec{c}\lambda$

1-27-71

FREE VECTORS VS. BOUND VECTORS

10 RULES FOR VECTOR SPACE

$\vec{a}+\vec{b} \in V$; $k\vec{a} \in V$

1-29-71

VECTOR SPACE AXIOMS

A_0 $\forall \vec{a}, \vec{b} \in V \ \exists\ \vec{a}+\vec{b} \in V \ni \vec{a}+\vec{b}$ IS UNIQUE

A_4 $\forall \vec{a}, \vec{b} \in V : \vec{a}+\vec{b} = \vec{b}+\vec{a}$

A_2 $\forall a, \vec{b}, c \in V ; \vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$ ETC. (IN BOOK)

S_0 $\forall \vec{a} \in V ; k \in R \ \exists\ k\vec{a} \in V \ni k\vec{a}$ IS UNIQUE

Pg 112 A SUBSPACE IS OF A SPACE IFF $\alpha\vec{U} + \beta\vec{V} = \vec{W} \ni \vec{W} \in$ SUBSPACE

(PICKING $\alpha, \beta, \vec{U}$, & $\vec{V}$ ACCORDINGLY TO FIT $\exists$ AXIOMS)

HOW ABOUT PLANE THRU THE ORIGIN:

$a_1X_1 + a_2X_2 + a_3X_3 = 0$ $A_0, A_2 A_3 S_0$ (3 AXIOMS)

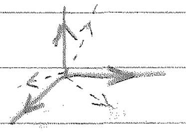
$a_1Y_1 + a_1Y_2 + a_3Y_3 = 0$

$\vec{Z} = \alpha\vec{X}_1 + \beta\vec{Y}_1$

2-3-71

VECTOR SPACE: A, S

NORMED VECTOR SPACE: D



DIMENSION = # OF UNIT VECTORS

LINEAR DEPENDENCE IS NEAT

Ex.) $\vec{a} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{d} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{e} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 2 \end{bmatrix}$ $f = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

ARE THESE VECTORS DEFINE A SUBSPACE,

$\vec{a}\,\vec{b}\,\vec{c} \in S$ $\vec{d}\,\vec{e}\,\vec{f} \in T$ FIND $S \cap T$

SHOW S IS LINEARLY INDEPENDENT:

$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ SHOULD NOT BE ABLE TO SOLVE ('CEPT $\vec{\alpha} = \vec{0}$)

$\alpha_1 \vec{a} + \alpha_2 b + \alpha_3 c = 0$ IND. IF ONE α_n IS NON-ZERO

ie) IN 3 DIMENSIONS 3 VECTORS (UNIT) CANNOT DEFINE SPACE IF IN SAME PANE

$$\begin{bmatrix} 2 & 1 & 0 & 3 & 1 & 2 \\ 3 & 1 & 1 & -1 & -3 & -1 \\ 0 & 1 & -1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ -\beta_1 \\ -\beta_2 \\ -\beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \overset{\text{REF}}{\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & -\beta_1 & -\beta_2 & -\beta_3 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -4 & -1 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}}$$

$\alpha_1 = \beta_2$

$\alpha_2 = -4\beta_2 - \beta_3$ $\vec{w} = \alpha_1 a + \alpha_2 b_2 + \alpha_3 \vec{c} = \beta_1 \vec{d} + \beta_2 e + \beta_3 \vec{f}$

$\alpha_3 = -3\beta_2 - \beta_3$ ~~α_1~~ ETC.

$\beta_1 = -\beta_2 - \beta_3$ $\Rightarrow \beta_3(\vec{a} - 4\vec{b} - 3\vec{c} - \beta_3(\vec{b} + \vec{c}) = S \cap T$

2-4-71

$\vec{a} = (2\,3\,0,1)^T$; $\vec{b} = (1,1,1,0)^T$ $\vec{c} = (0,1,{}^-1,0)^T$

$\vec{d}\,(3,1,2,1)^T$ $\vec{e} = (1,{}^-3,1,2)^T$ $f = (2,{}^-1,2,1)^T$

$\vec{a},\vec{b},\vec{c}$ FORM SUBSET $S \in R_4$

$\vec{d},\vec{e},f$ FORM SUBSET $T \in R_4$

CHECK IF $\vec{a},\vec{b},\vec{c}$ ARE LINEARLY INDEPENDENT

CHECK " $\vec{e},\vec{f},\vec{g}$ " " "

IF SO, $S = S_3$; $T = T_3$

$$\vec{W} = \alpha_1 \vec{a} + \alpha_2 \vec{b} + \alpha_3 \vec{c}$$
$$= \beta_1 \vec{d} + \beta_2 \vec{e} + \beta_3 \vec{f}$$

$$\begin{bmatrix} 2 & 1 & 0 & 3 & 1 & 2 \\ 3 & 1 & 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{REF} \begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & -\beta_1 & -\beta_2 & -\beta_3 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & {}^-4 & -1 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

$$\Rightarrow \alpha_1 - \beta_1 = 0$$
$$\alpha_2 + \beta_2 + \beta_3 = 0$$
$$\alpha_3 + 3\beta_2 + 4\beta_3 = 0$$
$$\beta_1 + \beta_2 + \beta_3 = 0$$

$$\Rightarrow \vec{W} = \beta_2 a - 4\beta_2 \vec{b} - \beta_3 \vec{b} - 3\beta_2 \vec{c} - \beta_3 \vec{c}$$
$$= {}^-\beta_2(\vec{-a} + 4\vec{b} + 3\vec{c}) - \beta_3(\vec{b} + \vec{c})$$ → LOOKS LIKE VECTOR SPACE
$$= {}^-\beta_2 \vec{d} - \beta_3 \vec{d} + \beta_2 \vec{e} + \beta_3 \vec{f}$$
$$= \beta_2(-\vec{d} + \vec{e}) - \beta_3(\vec{d} - \vec{f})$$

ALSO $+d - e = \vec{a} - 4\vec{b} - 3\vec{c}$; $\vec{d} - \vec{f} = \vec{b} + \vec{c}$

$Y =$ $\vec{d} - \vec{e} = (2,4,1,{}^-1)$ | $\vec{b} + \vec{c} = (1,2,0,0)^T = X$

$-\vec{a} - 4\vec{b} + 3\vec{c} = (2,4,1,{}^-1)$ | $\vec{d} - \vec{f} = (1,2,0,0)^T$

$$\Rightarrow Z = X + Y$$

BASIS $(\vec{X},\vec{Y})$ OR $(\vec{X},\vec{Z})$

OF #'S IN VECTOR SPACE = # OF DEMINTION

RANK = SPACE OF ROW VECTORS

2-3-71

TEST TOMORROW, 4.5, 5.1-5.6

LECTURE:

$AX = 0$; HOMOGENEOUS | $Ax = b$; NON-HOMOGENEOUS

THE SET OF SOLUTIONS OF $A\vec{x} = \vec{b}$ IS NOT A VECTOR SPACE, FOR THERE IS NO $\vec{0}$

$$A\vec{x} = \vec{b}$$

$$\overset{A \rightarrow}{\begin{bmatrix} c_1 & c_2 & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{bmatrix}} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} = x_1 \begin{bmatrix} c_1 \\ \vdots \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} c_2 \\ \vdots \\ \vdots \end{bmatrix} + \cdots + x_n \begin{bmatrix} c_n \\ \vdots \\ \vdots \end{bmatrix}$$

THE ROW RANK OF A EQUALS THE DIMENSION OF THE ROWSPACE

COLUMN RANK OF A EQUALS THE DIMENSION OF THE (COUGH, HACK, WHEEZE) OF THE COLUMN SPACE

THEM: ROW RANK = COLUMN RANK

EX) $\overset{\text{REG}}{P}A$; SHOW $rr(PA) = rr(A)$

$$PA = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{matrix} r_1 \\ r_2 \end{matrix} = \begin{bmatrix} p_1 a_1 + q_1 a_2 & p_1 b_1 + q_1 b_2 \\ p_2 a_2 + q a_2 & p_2 b_1 + q_2 b_2 \end{bmatrix}$$

$$= \begin{bmatrix} p_1 r_1 + q_1 r_2 \\ p_2 r_1 + q_2 r_2 \end{bmatrix}$$ (LINEAR COMBINATIONS OF A)

LET $rr_A = S$; A LINEAR COMBINATION CAN-NOT INTRODUCE NEW VECTOR

$\Rightarrow rr(PA) \leq rr(A)$

$\because$ P IS NON-SINGULAR, P^{-1} IS NON-SINGULAR

$rr(P^{-1}PA) \leq rr(P^{-1}A)$

$rr(A) \leq rr(P^{-1}A) = rr(PA)$

$\Rightarrow rr(A) = rr(PA)$

HOW 'BOUT cr (PA);

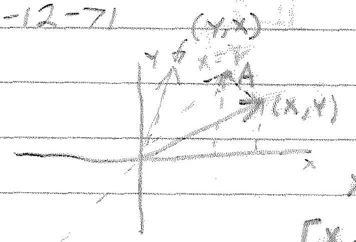
$$\lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda_2 \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \lambda_3 \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \lambda_4 \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = 0$$

LOTS OF MESSING AROUND YIELDS:

$rr(PA) = rr(A)$; $cr(PA) = cr(A)$

$$\begin{bmatrix} 0 & 1 & 0 & p & 0 & r \\ 0 & 0 & 1 & q & 0 & s \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ AS MANY "1"'S AS ROW RANK VOLUMN

2-12-71



FIND REF ON X=Y LINE

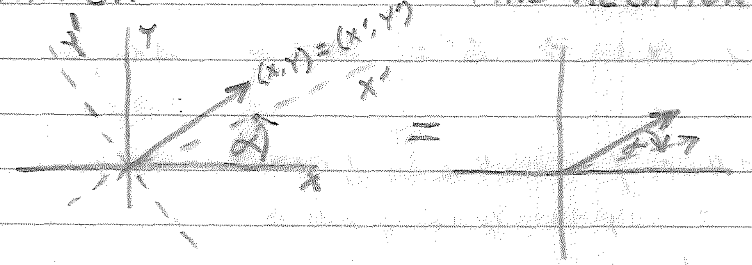
$\Rightarrow A = \left(\frac{X+Y}{2}, \frac{X+Y}{2} \right)$

$X_1 = \frac{1}{2} X + \frac{1}{2} Y$. $Y_1 = \frac{1}{2} X + \frac{1}{2} Y$

$$\Rightarrow \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

ROTATION

FIND RELATION OF X, X' ÷ Y, Y'

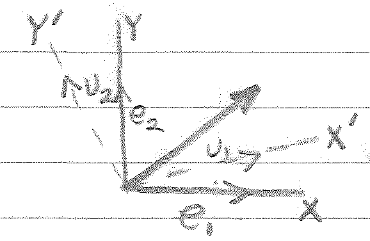


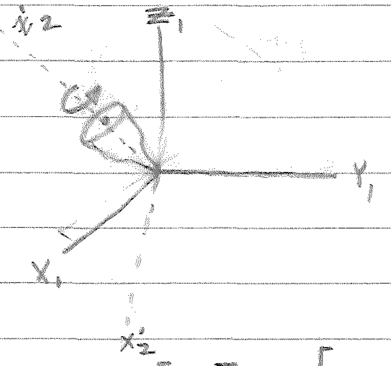
$$\vec{V}' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \vec{V}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

XOY; ORIGINAL COORDINATE SYSTEM

X'OY'; NEW " "





$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

$$\begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

$$\begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_3 \\ Y_3 \\ Z_3 \end{bmatrix}$$

(BY ROTATION:

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$)

$$\begin{bmatrix} X_4 \\ Y_4 \\ Z_4 \end{bmatrix} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta & \cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta \\ -\sin\psi\cos\phi - \cos\psi\sin\phi\sin\theta & -\sin\psi\cos\phi + \cos\psi\sin\phi\sin\theta \\ \sin\phi\sin\theta & \end{bmatrix}$$

ETC.

$$\Rightarrow \vec{V}_4 = CBA\vec{V}_1 \Rightarrow V_1 = \underbrace{A^{-1}B^{-1}C^{-1}}_{} \vec{V}_4$$

CHANGE ANGLE SIGNS ⇒ A ⇒ A^{-1}

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad ; \quad \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

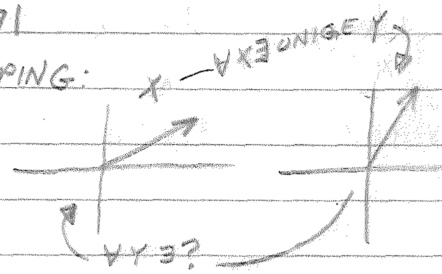
VECTOR NEW COOR. — SAME VECT OLD COOR — VECTOR O.C. — DISPLACED O.C.

$$\underset{\text{VECTOR N.C.}}{\begin{bmatrix} X_2' \\ Y_2' \\ Z_2' \end{bmatrix}} = P \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = PA \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = PAP^{-1} \begin{bmatrix} X_1' \\ Y_2' \\ Z_1' \end{bmatrix}$$

SIMILAR: $B = PAP^{-1}$

2-15-71

MAPPING:



ONE TO ONE CORRES.
IN VECTOR SPACES

CHAPT. 7

$\dot{X} = 2X - Y \;;\; \dot{Y} = 3X - 2Y$

$$\vec{X} = \begin{bmatrix} X \\ Y \end{bmatrix} ; \quad \vec{A} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \Rightarrow \dot{\vec{X}} = A\vec{X}$$

$$X = ue^{\lambda t} \Rightarrow \dot{X} = \lambda u e^{\lambda t} = \lambda X$$

$$Y = ve^{\lambda t} \Rightarrow \dot{Y} = \lambda v e^{\lambda t} = \lambda Y$$

$$\therefore \dot{\vec{X}} = \vec{U}\lambda e^{\lambda t} = \lambda\vec{X}$$

$$\Rightarrow \dot{\vec{X}} = \lambda\vec{U}e^{\lambda t} = \lambda\vec{X} \Rightarrow \lambda\vec{X} = \vec{A}\vec{X}$$

$$(A - \lambda I)\vec{X} = 0$$

PROBLEM: SOLVE FOR $\vec{X}$ $(A - \lambda I)\vec{X} = 0$, WITH A SQUARE

$$\vec{A} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \qquad \begin{bmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

SOLUTION $\vec{X} = 0$ (TRIVIAL)

TO HAVE SOLUTION, A MUST BE SINGULAR

$$\Rightarrow \det(A - \lambda I) = 0$$

(CONT.)

NEED ALL $\lambda \ni det(\vec{A} - \lambda\vec{I}) = \vec{0}$ WHICH IS A POLYNOMIAL OF DEGREE n IN λ CALLED CHARACTERISTIC (POLYNOMIAL) EQUATION.

$$\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

THE ROOTS OF THE CHARACTERISTIC EQUATION ARE CALLED EIGENVALUES

$$\lambda = 1 \Rightarrow \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad EV = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{x}$$

$$\lambda = -1 \Rightarrow \begin{bmatrix} +3 & -1 \\ +3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad EV = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \vec{x}$$

FOR EACH λ, $\exists$ A SEPARATE ANSWER SPACE

2-17-71

FINAL: WED 1:15 IN B119

LECTURE:

CHARACTERISTIC EQUATION: $det(A - \lambda I)$

Pg 217

1) g) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\lambda_1 = -1$; $\lambda_2 = 5$

$$\begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (-1-\lambda)^2(5-\lambda) \Rightarrow \lambda = -1 \text{ IS DOUBLE RT.}$$

SOLVE $(A - \lambda I)\vec{x} = \vec{0}$

$$\lambda_1 = -1 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{bmatrix} \lambda \\ \mu \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mu\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 5 \Rightarrow \begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{x} = \vec{0} \Rightarrow \vec{x} = \alpha\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

ALL EIGEN VALUES'
VECTOR SPACES

$\forall\ A_{nn}$, # EIGEN VECTORS $\leq n$

$A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x}$

$\therefore A^n\vec{x} = \lambda^n\vec{x}$

$B = P^{-1}AP$ IF $B \overset{SIM}{\sim} A$

$$\begin{aligned}\det(B-\lambda I) &= \det(P^{-1}AP-\lambda I) = \det(P^{-1}AP - \lambda P^{-1}P)\\ &= \det\left[P^{-1}(A-\lambda I)P\right]\\ &= \det P^{-1}\det(A-\lambda I)\det P\\ &= \det P^{-1}\det P\det(A-\lambda I)\\ &= \det\left[P^{-1}P\right]\det(A-\lambda I) = \det(A-\lambda I)\end{aligned}$$

FIND $P \ni P^{-1}AP = D \ni D$ IS IN DIAGONAL FORM

LET $P = \begin{bmatrix} x_1 & x_2 & x_3 \cdots \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot \end{bmatrix}$ WHICH ARE EIGEN VECTORS

$$\begin{aligned}\therefore\ P^{-1}A\left[\vec{x}_1\,\vec{x}_2\ldots\vec{x}_n\right] &= P^{-1}\left[A\vec{x}_1\ A\vec{x}_2\ldots A\vec{x}_n\right]\\ &= P^{-1}\left[\lambda_1\vec{x}_1\ \lambda_2\vec{x}_2\ldots\lambda_n\vec{x}_n\right]\end{aligned}$$

$$\left(P^{-1}\left[\vec{x}_1\vec{x}_2\ldots\vec{x}_n\right] = I\right) \quad = \left[\lambda_1 \underset{\uparrow}{\,} P^{-1}\vec{x}_1\ \ \lambda_2 P^{-1}x^2 \ldots \lambda_n P^{-1}x^n\right]$$

$$D = \begin{bmatrix}\lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & \cdots & \\ | & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix} \Rightarrow \det(D-\lambda I) = (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n)$$

ALL $\vec{x} \in P$ MUST BE LINEARLY INDEPENDENT

THEM: EIGEN VALUES OF A REAL SYMMETRIC MATRIX ARE REAL $(A = A^T)$

2-22-71

CAYLEY-HAMILTON

IF $a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda_n^n = \det(A - \lambda I)$

THEN $a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = 0$

PROOF

$(A - \lambda I)\,adj\,(A - \lambda I) = I\det(A - \lambda I)$

$= a_0 I + a_1 \lambda I + \dots + a_n \lambda_n I =$

$(A - \lambda I)(B_0 + B_1\lambda + \dots + B_n\lambda^{n-1})$

$AB_0 + (AB_1 - B_0)\lambda + (AB_2 + B_1)\lambda^2 + \dots + (AB_{n-1} + B_{n-2})\lambda^{n-1}$

$\dots + B_{n-1}\lambda^n$

$\Rightarrow AB_0 = a_0 I \qquad\qquad AB_0 = a_0 I$

$AB_1 - B_0 = a_1 I \qquad\qquad A^2B_1 - AB_0 = a_1 A$

$AB_2 - B_1 \pm a_2 I \qquad\qquad A^3B_2 - A^2B_1 = a_2 A^2$

$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$

$AB_{n-1} - B_{n-2} = a_{n-1} I \qquad\qquad \Rightarrow a_0 I + a_1 A + \dots + a_n A^n$

$-B_{n-1} = a_n I$

EX) SUPPOSE CHAR. EQ. OF A IS $\lambda^3 - 2\lambda^2 + 3\lambda - 1 = 0$

THE ACCORDING TO C-H:

$A^3 - 2A^2 + 3A - I = 0$

$\Rightarrow A^2 - 2A + 3I = A^{-1}$

$e^A = I + A + \frac{1}{2}A^2 + \dots + \frac{1}{n!}A^n + \dots$

$\sin A = I - \frac{A^3}{3!} + \frac{A^5}{5!} + \dots$

$S_0 = I \qquad\qquad S_2 = 1 + A + \frac{1}{2}A^2$

$S_1 = I + A \qquad\qquad S_3 = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3$

$S_n = \sum_{n=0}^{n} \frac{A^n}{n!}$

$\dot{x} = Ce^{At}$

PROB: IF A HAS CHAR. EQ $\lambda^3 - 2\lambda^2 + 3\lambda - 1 = 0$

EXPRESS THE PARTIAL SUMS $S_0, S_1, \dots S_r$

AS QUADRATIC POLYNOMIAL

IF A HAS MULTIPLE EIGEN VALUES, THIS CHAR. EQUATION CAN BE WRITTEN:

$$(\lambda-\lambda_1)^{\alpha_1}(\lambda-\lambda_2)^{\alpha_2}\cdots(\lambda-\lambda_s)^{\alpha_s}$$

C-H SAYS $(A-\lambda_1 I)^{\alpha_1}(A-\lambda_2 I)^{\alpha_2}\cdots(A-\lambda_s I)^{\alpha_s}=0$

IT MAY HAPPEN THAT A SATISFIES

$$(A-\lambda_1 I)^{\beta_1}(A-\lambda_2 I)^{\beta_2}\cdots(A-\lambda_s I)^{\beta_s}=0$$

$\Rightarrow \beta_1 \le \alpha_1;\ \beta_2 \le \alpha_2$ ETC.

THIS IS CALLED THE MINIMUM EQUATION

EX) $A=\begin{bmatrix}0 & 0 & 1\\ 8 & 0 & 2\\ 0 & -1 & 5\end{bmatrix}$ $P(\lambda) = -8-2\lambda+5\lambda^2-\lambda^3=0$

$\Rightarrow \lambda = (-1, 2, 4)$

CH. eq $\Rightarrow$ $+8I+2A-5A^2+A^3=0$

EX) $A=\begin{bmatrix}7 & 4 & -1\\ 4 & 7 & -1\\ -4 & -4 & 4\end{bmatrix}$ $\Rightarrow P(\lambda) = 108-81\lambda+18\lambda^2-\lambda^3$

$\lambda = (3, 3, 12)$

$P(\lambda) = -(\lambda-3)^3(\lambda-12)$

CH eq $\Rightarrow (A-3I)^2(A-12I)=0$

$\therefore (A-3I)(A-12I)\overset{?}{=}0$ YES!

EX) $A=\begin{bmatrix}2 & -2 & 3\\ 10 & -4 & 5\\ 5 & -4 & 6\end{bmatrix}$ $\Rightarrow P(\lambda) = -(\lambda-1)^2(\lambda-2)$

$= 2-5\lambda+4\lambda^2-\lambda^3$

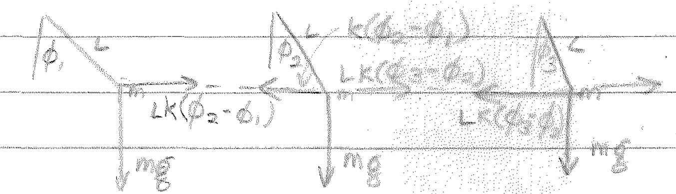
CH $\Rightarrow A^3-4A^2+5A-2I=0$

$(A-I)(A-2I) = A^2-3A+2I \neq I$

PROB) $A=\begin{bmatrix}2 & 1 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & -2 & 4\end{bmatrix}$ $P(\lambda) = (\lambda-2)^3(\lambda-3)$

WHAT IS MINIMUM EQ?

2-23-71



$$\begin{cases} mL^2\ddot{\phi}_1 = [k(\phi_2-\phi_1) - mg\phi_1]L \\ mL^2\ddot{\phi}_2 = L[Lk(\phi_3-\phi_2) - Lk(\phi_2-\phi_1)] - mg\phi_2 L \\ mL^2\ddot{\phi}_3 = -L[kL(\phi_3-\phi_2) - mgL\phi_3 \end{cases}$$

$$\ddot{\phi}_1 = -\left(\frac{k}{m}+\frac{g}{L}\right)\phi_1 + \frac{k}{m}\phi_2$$

$$\ddot{\phi}_2 = \frac{k}{m}\phi_1 - \left(\frac{2k}{m}+\frac{g}{L}\right)\phi_2 + \frac{k}{m}\phi_3$$

$$\ddot{\phi}_3 = \frac{k}{m}\phi_2 - \left(\frac{k}{m}+\frac{g}{L}\right)\phi_3$$

FIND 3 EIGEN VALUES

(FIND EIGEN VALUES)

LEGRANGE-SYLVESTER

$$F(A) = C_{n-1}A^{n-1} + C_{n-2}A^{n-2} + \dots + C_0 I$$

$$(\det(A-\lambda I) = P(\lambda) \Rightarrow P(\lambda_r) = 0 \Rightarrow \lambda_r = \text{EIGEN VALUES})$$

$$F(\lambda_1) = C_{n-1}\lambda_1^{n-1} + C_{n-2}\lambda_1^{n-2} + \dots + C_0$$

$$F(\lambda_2) = C_{n-1}\lambda_2^{n-1} + C_{n-2}\lambda_2^{n-2} + \dots + C_0$$

$$\vdots$$

$$F(\lambda_n) = C_{n-1}\lambda_n^{n-1} + C_{n-2}\lambda_n^{n-2} + \dots + C_0$$

$$\begin{bmatrix} F(A) & A^{n-1} & \cdots & A & I \\ F(\lambda_1) & \lambda_1^{n-1} & \cdots & \lambda_1 & 1 \\ \vdots & & & \vdots & \vdots \\ F(\lambda_n & \lambda_n^{n-1} & \cdots & \lambda_n & 1 \end{bmatrix} = 0 \qquad D_0 = \begin{bmatrix} \lambda_1^{n-1} & \cdots & \lambda_1 & 1 \\ \lambda_2^{n-1} & \cdots & \lambda_2 & 1 \\ \lambda_n^{n-1} & \cdots & \lambda_n & 1 \end{bmatrix}$$

$$F(A) = L_1(A)F(\lambda_1) + L_2(A)F(\lambda_2) + \dots + L(A)F(\lambda_n)$$

$$D_1 = \begin{vmatrix} A^{n-1} & \cdots & A & I \\ \lambda_2^{n-1} & \cdots & \lambda_2 & 1 \\ \lambda_n^{n-1} & & \lambda_n & 1 \end{vmatrix} \qquad \Rightarrow L_1 = D_1/D_0$$

$$D_0 = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)(\lambda_2-\lambda_2)\cdots(\lambda_2-\lambda_n)\cdots\lambda_{n-1}$$

$D_1 \Rightarrow$ REPLACE ~~A_n~~ λ_1 BY A_1

$D_n \Rightarrow$ " λ_n BY A_n

$$L_1 = \frac{D_1}{D_0}$$

$L_1 = \frac{(A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_n I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_n)}$ L-LAGRANGIAN COEFFICIENT

$L_2 = \frac{(A-\lambda_1 I)(A-\lambda_3 I)\cdots(A-\lambda_n I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_n)}$

EX) $\dot{x} = Ax \Rightarrow x(0) = C$

$\Rightarrow x = e^{At}C$

$e^{At} = L_1(A)e^{\lambda_1(t)} + L_2(A)e^{\lambda_2 t} + \ldots + L_n(A)e^{\lambda_n t}$

2-24-71

$\dot{x} = Ax \Rightarrow x = e^{At}C \Rightarrow e^{At} = I + At + \frac{A^2t^2}{2!} + \ldots + \frac{A^n t^n}{n!}$

$F(A) = L_1(A)F(\lambda_1) + L_2(A)F(\lambda_2) + \ldots + L_n(A)F(\lambda_n)$

$n=3 \Rightarrow L_1 = \frac{(A-\lambda_2 I)(A-\lambda_3 I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$; $L_2 = \frac{(A-\lambda_1 I)(A-\lambda_3 I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}$

$L_3 = \frac{(A-\lambda_1 I)(A-\lambda_2 I)}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}$

$\Rightarrow e^{At} = L_1(A)e^{\lambda_1 t} + L_2(A)e^{\lambda_2 t} + L_3(A)e^{\lambda_3(t)}$

$\therefore X = [L_1(A)e^{\lambda_1 t} + L_2(A)e^{\lambda_2 t} + L_3(A)e^{\lambda_3 t}]C$

$\dot{x} = Ax + f(t)$

$\dot{x}_c = Ax_c \Rightarrow x_c = e^{At}C$

$x_p = e^{At}v(t) \Rightarrow \dot{x}_p = Ae^{At}v(t) + e^{At}v'(t)$

$Ae^{At}v(t) + e^{At}\dot{v}(t) = e^{At}v(t) + f(t)$

$\Longrightarrow e^{At}\dot{v}(t) = f(t)$

$\therefore \dot{v} = e^{-At}f(t)$

$\Rightarrow v = \int e^{-At}f(t)dt$; $x_p = e^{At}\int e^{-At}f(t)dt$

$$\left[\begin{array}{l} x_p = L_1(A)e^{At}\int e^{-\lambda_1 t}f(t)dt + L_2(A)e^{+\lambda_2 t}\int e^{-\lambda_2 t}f(t)dt \\ \quad + L_3(A)e^{\lambda_3 t}\int e^{-\lambda_3 t}f(t)dt \end{array}\right.$$

2-26-71

QUIZ

2) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix} \Rightarrow$ CHAR. EQ $\Rightarrow \lambda^2(\lambda-6)=0$

FOR $\lambda=0$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = 0 \Rightarrow U_1+U_2-2U_3=0$$

$$\alpha\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + B\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

FOR $\lambda=6$

$$\lambda = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

1) $\begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow (\lambda-2)^2(\lambda-5)=0$

FOR $\lambda=2$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ FOR $\lambda=5$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$(A-\lambda I)U=0 \ni (A-\lambda I=0); U\neq\vec{0}$

3) Pg 218-19

TAKE EIGEN VECTORS & PUT IN A MATRIX P

$$P = \begin{array}{c} \begin{array}{ccc} \lambda=2 & 2 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

FINAL HINTS

EIGEN VALUES & VECTORS

CHAR. EQ. KNOW L'S $\left(L_K = \frac{(A-\lambda_1 I)(A-\lambda_2 I)\ldots(}{(\lambda_K-\lambda_1)\ldots}\right)$

SIMILARITY PROBLEM; LOOK OVER OLD TESTS

Linear Algebra

The first test will cover the following:

Chapter 2

Addition and multiplication of matrices.

Writing a set of simultaneous linear algebraic equations in matrix notation.

To find the inverse of a 2x2 matrix (p. 29)

To use the inverse two solve two equations in two unknowns (p. 34)

Chapter 3

Solving a system of equations by means of the augmented matrix (p. 40-41)

To show that two matrices A and B are row equivalent by constructing a product P of elementary matrices such that B = PA (p. 47)

To reduce a matrix to REF by row operations (p. 49-53)

Pg 47

Pg 21 12-9-70

1) $A+B=B+A=\begin{bmatrix} 3 & 2 & 4 \\ 11 & 6 & 8 \\ -2 & 1 & -1 \end{bmatrix}$

$C+D=D+C=\begin{bmatrix} 3 & 9 & 0 & 1 \\ 1 & 7 & 1 & 2 \\ 6 & 3 & 3 & -6 \end{bmatrix}$

$C+F=F+C=\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix}$

$F+D=D+F=\begin{bmatrix} 2 & 9 & 0 & 9 \\ 2 & 6 & 1 & 9 \\ 6 & 2 & 2 & -7 \end{bmatrix}$

2) a) $3A=\begin{bmatrix} 6 & 3 & 9 \\ 27 & 12 & 18 \\ -3 & 6 & 0 \end{bmatrix}$

b) $-2B=\begin{bmatrix} -2 & -2 & -2 \\ -4 & -4 & -4 \\ 2 & 2 & 2 \end{bmatrix} \rightarrow A-2B=\begin{bmatrix} 0 & -1 & 1 \\ 5 & 0 & 2 \\ 1 & 4 & 2 \end{bmatrix}$

c) $2C+5D$

$2C=\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$ $5D=\begin{bmatrix} 10 & 45 & 0 & 0 \\ 5 & 30 & 5 & 5 \\ 30 & 15 & 10 & -35 \end{bmatrix}$

$\Rightarrow 2C+5D=\begin{bmatrix} 12 & 45 & 0 & 2 \\ 5 & 32 & 5 & 7 \\ 30 & 15 & 12 & -33 \end{bmatrix}$

d) $-G=\begin{bmatrix} -2 & 0 & -4 \\ -9 & -1 & -5 \end{bmatrix}$

e) $C+D-F=\begin{bmatrix} 3 & 9 & 0 & 1 \\ 0 & 7 & 1 & 2 \\ 6 & 4 & 3 & -6 \end{bmatrix}$

Pg 25 12-14-70

1) a) $5\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}5 & 10\\15 & 20\end{bmatrix}$

b) $\begin{bmatrix}5 & 0\\0 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} = \begin{bmatrix}5 & 10\\15 & 20\end{bmatrix}$

c) $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}4+10+18\end{bmatrix} = \begin{bmatrix}32\end{bmatrix}$

d) $\begin{bmatrix}4\\5\\6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix} = \begin{bmatrix}4 & 8 & 12\\5 & 10 & 15\\6 & 12 & 18\end{bmatrix}$

e) $\begin{bmatrix}1 & 0 & 3\\2 & -1 & 0\end{bmatrix}\begin{bmatrix}7\\-3\\2\end{bmatrix} = \begin{bmatrix}13\\17\end{bmatrix}$

f) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}18 & 72\\-3 & 5\end{bmatrix} = \begin{bmatrix}18 & 72\\-3 & 5\end{bmatrix}$

g) $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

h) $\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

i) $\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}1 & 3\\-3 & 1\end{bmatrix}$

j) $\begin{bmatrix}2 & 3\\-3 & 2\end{bmatrix}\begin{bmatrix}8 & 2\\-2 & 8\end{bmatrix} = \begin{bmatrix}10 & 28\\-28 & 10\end{bmatrix}$

k) $\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

l) $\begin{bmatrix}0 & 1\\0 & 3\end{bmatrix}\begin{bmatrix}4 & 7\\0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

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7)

$$A^2 = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 4 \\ 7 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 8 & -4 & 4 \\ 7 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 20 & -12 & 12 \\ 19 & -11 & 15 \\ 7 & -7 & 8 \end{bmatrix}$$

$$-5A^2 = \begin{bmatrix} -40 & 20 & -20 \\ -35 & +15 & -20 \\ -15 & 15 & -20 \end{bmatrix}$$

$$8A = \begin{bmatrix} 24 & -8 & 8 \\ 16 & 0 & 8 \\ 8 & -8 & 16 \end{bmatrix}$$

$$-4I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 20 & -12 & 12 \\ 19 & -11 & 15 \\ 7 & -7 & 8 \end{bmatrix} + \begin{bmatrix} -40 & 20 & -20 \\ -35 & 15 & -20 \\ -15 & 15 & -20 \end{bmatrix} + \begin{bmatrix} 24 & -8 & 8 \\ 16 & 0 & 8 \\ 8 & -8 & 16 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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6) a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$1 = a_{11}$ $0 = a_{12}$ $0 = a_{21}$ $1 = 2a_{22} \Rightarrow a_{22} = \frac{1}{2}$

$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$1 = a_{11}$ $0 = a_{12}$ $0 = a_{13}$
$0 = a_{21}$ $a_{22} = 1/2$ $a_{23} = 0$ $\Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$
$0 = a_{31}$ $0 = a_{32}$ $a_{33} = \frac{1}{3}$

c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$a_{11} + 2a_{21} + 3a_{31} = 1$
$a_{12} + 2a_{22} + 3a_{32} = 0$
$a_{13} + 2a_{23} + 3a_{33} = 0$ ETC.

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EX. 3) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right]$ $R_3 \to R_1 - R_3$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & +1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$ $\Rightarrow \infty$ # OF SOLUTIONS

3) a) $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right]$ $R_2 \to 2R_1 - R_2$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right]$ $\Rightarrow X_1 = 0 \ ; X_2 = 0$

b) $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right]$ $R_2 \to 2R_1 - R_2$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right]$ $\Rightarrow X_2 = 0 \ ; X_1 = 0$

c) $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right]$ $R_2 \to 2R_1 - R_2$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\Rightarrow$ INFINITE # OF SOLUTIONS

d) $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 2 \end{array}\right]$ $R_2 \to 2R_1 - R_2$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & -2 \end{array}\right]$ $\Rightarrow$ NO SOLUTIONS

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6) a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 24 & 0 & 0 \\ 12 & 6 & 0 \\ -2 & 5 & 4 \end{bmatrix} \Rightarrow_2 \begin{bmatrix} 24 & 12 & -2 \\ 0 & 6 & 5 \\ 0 & 0 & 4 \end{bmatrix}\frac{1}{24} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{12} \\ 0 & 1/4 & 5/24 \\ 0 & 0 & 1/6 \end{bmatrix}$

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1) a) $5\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}$

b) $\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} = \begin{bmatrix}5 & 10\\ 15 & 20\end{bmatrix}$

c) $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}4\\ 5\\ 6\end{bmatrix} = \begin{bmatrix}32\end{bmatrix}$

d) $\begin{bmatrix}4\\ 5\\ 6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix} = \begin{bmatrix}4 & 8 & 12\\ 5 & 10 & 15\\ 6 & 12 & 18\end{bmatrix}$

f) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}18 & 73\\ -3 & 5\end{bmatrix} = \begin{bmatrix}18 & 73\\ -3 & 5\end{bmatrix}$

e) $\begin{bmatrix}1 & 0 & 3\\ 2 & -1 & 0\end{bmatrix}\begin{bmatrix}7\\ -3\\ 2\end{bmatrix} = \begin{bmatrix}13\\ 17\end{bmatrix}$

g) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

h) $\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

i) $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ -2 & 2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -4 & 1\end{bmatrix}$

j) $\begin{bmatrix}2 & 3\\ -3 & 2\end{bmatrix}\begin{bmatrix}8 & 2\\ -2 & 8\end{bmatrix} = \begin{bmatrix}10 & 28\\ -28 & 10\end{bmatrix}$

k) $\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

l) $\begin{bmatrix}0 & 1\\ 0 & 3\end{bmatrix}\begin{bmatrix}4 & 7\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

2) $\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn}\end{bmatrix}\begin{bmatrix}O_{11} & \cdots & O_{1p}\\ O_{12} & & \vdots\\ \vdots & & \\ O_{n1} & \cdots & O_{np}\end{bmatrix} = \begin{bmatrix}O_{11} & \cdots & O_{1p}\\ \vdots & & \vdots\\ O_{m1} & \cdots & O_{mp}\end{bmatrix}$

$O_{mn}A_{np} = O_{mp}$

$= A_{mn}O_{np} = O_{mp}$

3) $\begin{bmatrix}a_{11} & a_{12} & a_{13} & \cdots & a_{1n}\\ a_{21} & & & & \\ a_{31} & & & & \\ \vdots & & & & \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & b_{13} & \cdots & b_{1n}\\ b_{21} & & & & \\ b_{31} & & & & \\ \vdots & & & & \\ b_{n1} & \cdots & \cdots & \cdots & b_{nn}\end{bmatrix} = \begin{bmatrix}a_{11}b_{11} + a_{12}b_{21} + \cdots\\ a_{21}b_{11} + \\ \\ \end{bmatrix}$

NO!

4) b) $1 + 3i$; $5 + i$; $+i$; 1

c) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

c) $\begin{bmatrix}1 & 3\\ -3 & 1\end{bmatrix}\begin{bmatrix}5 & 1\\ -1 & 5\end{bmatrix} = \begin{bmatrix}2 & 16\\ -16 & 2\end{bmatrix}$

$(1+3i)(5+i) = 5 + i + 15i - 3 = 2 + 16i \Rightarrow \begin{bmatrix}2 & 16\\ -16 & 2\end{bmatrix}$

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6) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 6 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 4 & 0 \\ 9 & 0 & 0 \end{bmatrix}$

7) $A^2 \Rightarrow \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 4 \\ 7 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$A^3 = \begin{bmatrix} 8 & -4 & 4 \\ 7 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 20 & -12 & 12 \\ 19 & -11 & 12 \\ 7 & -7 & 8 \end{bmatrix}$

$$\underset{A^3}{\begin{bmatrix} 20 & -12 & 12 \\ 19 & -11 & 12 \\ 7 & -7 & 8 \end{bmatrix}} + \underset{(-5A^2)}{\begin{bmatrix} -40 & 20 & -20 \\ -35 & 15 & -20 \\ -15 & 15 & -20 \end{bmatrix}} + \underset{8A}{\begin{bmatrix} 24 & -8 & 8 \\ 16 & 0 & 8 \\ 8 & -8 & 16 \end{bmatrix}} + \underset{(-4I)}{\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

8) $A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & \\ \vdots & \\ a_{m1} \cdots & a_{mn} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & \\ \vdots & \\ b_{m1} \cdots & b_{mn} \end{bmatrix}$ $C = \begin{bmatrix} c_{11} & c_{12} \cdots c_{1p} \\ c_{22} & \vdots \\ \vdots & \\ c_{n1} & c_{np} \end{bmatrix}$

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5) $(A_1 A_2 \cdots A_K)^{-1} = A_K^{-1} A_{K-1}^{-1} \cdots A_1^{-1}$

$(A_1 A_2 \cdots A_K)^{-1}(A_1 A_2 \cdots A_K) = I = (A_K^{-1} A_K)(A_{K-1}^{-1} A_{K-1}) \cdots (A_1^{-1} A_1)$

$= (A_K A_K^{-1})(A_{K-1}^{-1} A_{K-1}) \cdots (A_1 A_1^{-K})$

$(A_1 A_2 \cdots A_K)^{-1}(A_1 A_2 \cdots A_K)^{-1} = (A_K A_{K-1} \cdots A_1)^{-1}(A_K A_{K-1} \cdots A_1)$

ARG!

$(A_1 A_2 \cdots A_K)^{-1} = (A_2 A_3 \cdots A_K)^{-1} A_1^{-1}$

$= (A_3 A_4 \cdots A_K) A_2^{-1} A_1^{-1}$

$= (A_4 A_5 \cdots A_K) A_3^{-1} A_2^{-1} A_1^{-1}$

$= A_K^{-1} A_{K-1}^{-1} \cdots A_1^{-1}$

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3) a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

$R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 \to 2R_1 - R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_1 + R_2 + R_3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$R_2 \to 2R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$R_1 \to R_2$
$R_2 \to R_1$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$R_3 \to R_3 - R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \to R_1 - R_2$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & +1 \end{bmatrix}$

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20) a) $3x_1 + 4x_2 = 0$
$2x_1 + 3x_2 = 5$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$3W + 4Y = 1$ $\quad$ $3X + 4Z = 0$
$\underline{2W + 3Y = 0}$ $\quad$ $\underline{2X + 3Z = 1}$
$-Y = 2$ $\quad$ $Z = 3$
$W = 3$ $\quad$ $X = -4$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -20 \\ 15 \end{bmatrix}$$

b) $3x_1 + 2x_2 = -3$
$7x_1 + 5x_2 = 1$

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$7(3W + 2Y = 1)$ $\quad$ $7(3X + 2Z = 0)$
$3(7W + 5Y = 0)$ $\quad$ $3(7X + 5Z = 1)$
$-Y = 7$ $\quad$ $-Z = 3$
$W = 5$ $\quad$ $X = -2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -13 \\ 18 \end{bmatrix}$$

c) $7x_1 + 3x_2 = 12$
$9x_1 + 4x_2 = 7$

$$\begin{bmatrix} 7 & 3 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -9 & 7 \end{bmatrix} \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ -59 \end{bmatrix}$$

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1) a) $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 1 & -3 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & -3 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \to R_3]{R_3 \to R_2} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$\xrightarrow{R_2 \to -\frac{1}{3}R_2} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \to 3R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 5 & 7 & 9 \end{bmatrix} \xrightarrow{R_2 \to 2R_1 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 3 & 5 & 7 & 9 \end{bmatrix} \xrightarrow{R_3 \to 3R_1 - R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 2 \\ 9 & 5 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{4}R_1} \begin{bmatrix} 1 & \frac{1}{2} \\ 9 & 5 \end{bmatrix} \xrightarrow{R_2 \to 9R_1 - R_2} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \xrightarrow{R_2 \to -2R_2} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d) $\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \to -R_1} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_1 - R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

$\xrightarrow[R_3 \to R_2]{R_2 \to R_3} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$R_1 \longrightarrow \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ -3 & -6 & -9 \end{bmatrix} \xrightarrow{R_3 \to 3R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_1 + R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

$\xrightarrow{R_3 \to -\frac{1}{3}R^3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

g) $\begin{bmatrix} 3 & 2 & -3 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_1]{R_1 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ 3 & 2 & -3 & 1 \end{bmatrix} \xrightarrow{R_2 \to 2R_1 - R_2} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & 4 \\ 3 & 2 & -3 & 1 \end{bmatrix} \xrightarrow{R_3 \to 3R_1 - R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 6 & 5 \end{bmatrix}$

$\xrightarrow[R_3 \to R_1]{R_1 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 6 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix} \xrightarrow{R_3 \to 3R_2 - R_3} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 17 & 11 \end{bmatrix} \xrightarrow[R_3 \to \frac{1}{17}R_2]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 5 \\ 0 & 0 & 1 & 1/11 \end{bmatrix}$ ETC.

2) a) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{matrix} R_2 \to 2R_1 - R_2 \\ R_3 \to 3R_1 - R_3 \\ R_4 \to 4R_1 - R_4 \end{matrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 3 \\ 6 & 4 \end{bmatrix} R_2 \to R_1 + R_2 \to \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 3 \\ 6 & 4 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 3 \\ 6 & 3 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \\ 6 & 3 \end{bmatrix} R_3 \to R_3 - R_1 - 3R_2$

c) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 7 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to 3R_1 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & -1 \\ 7 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_3]{R_3 \to R_4} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \\ 7 & -2 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \\ 7 & -2 & 6 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & -2 & 6 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - 2R_2 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 7 & -2 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 4 \\ 9 & 4 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \to 9R_1 - R_2} \begin{bmatrix} 1 & 4 \\ 0 & 32 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{32}R_2} \begin{bmatrix} 1 & 4 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \to 4R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_1 - 4R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

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7) a) $\left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ -3 & 1 & 3 & 0 & 0 & 1 \end{array}\right] R_3 \to R_3 + R_1 \left[\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right] R_1 \to \frac{1}{3}R_1 \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right]$

$R_2 \to R_1 + R_2 \left[\begin{array}{ccc|ccc} 1 & 2/3 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 8/3 & 8/3 & \frac{1}{3} & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right] R_2 \to \frac{3}{8}R_2 \left[\begin{array}{ccc|ccc} 1 & 2/3 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 1/8 & 3/8 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right] R_3 \to \frac{1}{3}R_3 \left[\begin{array}{ccc|ccc} 1 & 2/3 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 1/8 & 3/8 & 0 \\ 0 & 1 & 2/3 & 1 & 0 & 1 \end{array}\right]$

$R_3 \to R_2 - R_3 \left[\begin{array}{ccc|ccc} 1 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 1/8 & 3/8 & 0 \\ 0 & 0 & 1/3 & -9/8 & 3/8 & -1 \end{array}\right] R_3 \to 3R_3 \left[\begin{array}{ccc|ccc} 1 & 2/3 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 1 & 1/8 & 3/8 & 0 \\ 0 & 0 & 1 & -27/8 & 9/8 & -3 \end{array}\right] R_2 \to R_2 - R_3 \left[\begin{array}{ccc|ccc} 1 & 2/3 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 & 28/8 & - & \\ 0 & 0 & 1 & -27/8 & 9/8 & \end{array}\right.$

$R_1 \to R_1 - \frac{2}{3}R_3$ ETC

d) $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] R_2 \to 3R_1 - R_2 \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & -1 \end{array}\right] R_1 \to R_1 - R_2 \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 2 & 3 & -1 \end{array}\right]$

$R_2 \to \frac{1}{2}R_2 \quad \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -\frac{1}{2} \end{array}\right]$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3/2 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

f) $\left[\begin{array}{cccc|cccc} 3 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_3 \to \frac{1}{7}R_3 \\ R_2 \to R_1 - R_2 \\ R_3 \to \frac{1}{6}R_3 \end{array} \left[\begin{array}{cccc|cccc} 3 & 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/6 \end{array}\right]$

$R_1 \to \frac{1}{3}R_1 \left[\begin{array}{cccc|cccc} 1 & 4/3 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/6 \end{array}\right] R_1 \to R_1 - 4/3 R_2 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 4/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/6 \end{array}\right]$

h) $\left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 6 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to 2R_1 - R_2 \\ R_3 \to 3R_1 - R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 0 & 11 & -4 & 2 & -1 & 0 \\ 0 & 11 & -4 & 3 & 0 & -1 \end{array}\right]$

$R_3 \to R_3 - R_1$ → SINGULAR → NO INVERSE

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2) a) $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 1 & -3 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{bmatrix} R_2 \to R_1 - R_2 \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$x_1 - x_4 = 0$
$x_2 - \frac{1}{3} x_4 = 0$
$x_3 - x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \\ 1 \\ 1 \end{bmatrix} x_4$$

c) $\begin{bmatrix} 4 & 2 \\ 9 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 \\ 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$

~~d)~~ $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_3$$

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2) a) $\begin{bmatrix} 1 & -3 & 0 & 2 \\ 1 & -3 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{bmatrix} R_2 \to R_1 - R_2 \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & +1 & -1 \\ 0 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

$x_1 - x_4 = 0$
$x_2 - \frac{1}{3} x_4 = 0$
$x_3 - x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \\ 1 \\ 1 \end{bmatrix} x_4$$

c) $\begin{bmatrix} 4 & 2 \\ 9 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 \\ 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2$

~~d)~~ $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_5$$

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2) b) $\left[\begin{array}{cc|c} 1 & 2 & 2 \\ -1 & -2 & -2 \\ 3/2 & 3 & 3/2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -3/2 \end{array}\right] \rightarrow$ INCONSISTANT

c) $\left[\begin{array}{ccc|c} 4 & 8 & 12 & 16 \\ 2 & 2 & -2 & -4 \\ 5 & -5 & 5 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 1 & -1 & -2 \\ 1 & -1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 1 & -1 & -2 \\ 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 3 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 1 & 3 & 3 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$

$X_1 = 1$
$X_2 = -3/2$
$X_3 = 3/2$

d) $\left[\begin{array}{ccc|c} 3 & 2 & -3 & 6 \\ 2 & -1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 3 & 2 & -3 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 6 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 6 & -3 \\ 0 & 3 & 1 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 17 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 6 & -3 \\ 0 & 0 & 1 & -\frac{6}{17} \end{array}\right]$

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1) a) $A=\begin{bmatrix}1&-1\\3&i\end{bmatrix}=(1)(i)-(-1)3=i+3$

c) $\begin{bmatrix}1&-2&5&0\\3&2&i&3\\0&0&3&-1\\0&0&0&4\end{bmatrix}=1\begin{bmatrix}2&i&3\\0&3&-1\\0&0&4\end{bmatrix}+2\begin{bmatrix}3&i&3\\0&3&-1\\0&0&4\end{bmatrix}+5\begin{bmatrix}3&2&3\\0&0&-1\\0&0&4\end{bmatrix}$

$=1\left[2\begin{vmatrix}3&-1\\0&4\end{vmatrix}-\cancel{i\begin{vmatrix}0&-1\\0&4\end{vmatrix}}+\cancel{3\begin{vmatrix}0&3\\0&0\end{vmatrix}}\right]+2\left[3\begin{vmatrix}3&-1\\0&4\end{vmatrix}-\cancel{i\begin{vmatrix}0&-1\\0&4\end{vmatrix}}+\cancel{3\begin{vmatrix}0&3\\0&0\end{vmatrix}}\right]$

$+5\left[\cancel{3\begin{vmatrix}0&-1\\0&4\end{vmatrix}}-\cancel{2\begin{vmatrix}0&-1\\0&4\end{vmatrix}}+\cancel{3\begin{vmatrix}0&0\\0&0\end{vmatrix}}\right]$

$=24+72=96$

e) $\begin{vmatrix}1&0&3\\2&0&4\\0&6&0\end{vmatrix}=\begin{vmatrix}0&4\\6&0\end{vmatrix}+3\begin{vmatrix}2&0\\0&6\end{vmatrix}=-24+36=12$

2) $\begin{vmatrix}1&0&0&0\\0&2&0&0\\0&0&3&0\\0&0&0&4\end{vmatrix}=1\begin{vmatrix}2&0&0\\0&3&0\\0&0&4\end{vmatrix}=2\begin{vmatrix}3&0\\0&4\end{vmatrix}=24$

7) a) $\begin{vmatrix}0&1&1\\0&1&2\\1&0&1\end{vmatrix}=\begin{vmatrix}1&1\\1&2\end{vmatrix}=1$

b) $\begin{vmatrix}1&2&2\\0&1&1\\1&0&1\end{vmatrix}=\begin{vmatrix}1&1\\0&1\end{vmatrix}+\begin{vmatrix}2&2\\1&1\end{vmatrix}=1$

c) $\begin{vmatrix}1&0&0&2\\0&2&1&1\\1&3&0&4\\2&0&0&2\end{vmatrix}=\begin{vmatrix}2&1&1\\3&0&4\\0&0&2\end{vmatrix}-2\begin{vmatrix}0&2&1\\1&3&0\\2&0&0\end{vmatrix}=2\begin{vmatrix}2&1\\3&0\end{vmatrix}-2\cdot2\begin{vmatrix}2&1\\3&0\end{vmatrix}=-6+12=6$

d) $\begin{vmatrix}1&6&2&-1\\-2&2&0&0\\4&4&0&0\\2&-3&1&5\end{vmatrix}=-1\begin{vmatrix}-2&2&0\\4&4&0\\2&-3&1\end{vmatrix}-5\begin{vmatrix}1&6&2\\-2&2&0\\4&4&0\end{vmatrix}=-1\left[-2\begin{vmatrix}4&0\\-3&1\end{vmatrix}-2\begin{vmatrix}4&0\\2&1\end{vmatrix}\right]-5\left[4\begin{vmatrix}6&2\\2&0\end{vmatrix}-4\begin{vmatrix}1&2\\-2&0\end{vmatrix}\right]$

$=+1[+8+8]+5[+16+16]=16$ ETC

f) $\begin{vmatrix}0&.5&0\\-.4&-.7&4\\.2&.3&3\end{vmatrix}=-.5\begin{vmatrix}-.4&4\\.2&3\end{vmatrix}=1$

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1) a) $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 7 \\ 6 & 2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 2 \\ 1 & 7 & 1 \end{bmatrix}$

e) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

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6) a) $\det A = \det\begin{bmatrix}2 & 4 & 0\\ 1 & 1 & 3\\ 2 & 1 & -1\end{bmatrix} = -\det\begin{bmatrix}1 & 1 & 3\\ 2 & 4 & 0\\ 2 & 1 & -1\end{bmatrix} = -\det\begin{bmatrix}1 & 1 & 3\\ 0 & -2 & 6\\ 0 & 1 & 7\end{bmatrix}$

$= \det\begin{bmatrix}1 & 1 & 3\\ 0 & 1 & 7\\ 0 & -2 & 6\end{bmatrix} = \det\begin{bmatrix}1 & 1 & 3\\ 0 & 1 & 7\\ 0 & 0 & 20\end{bmatrix} = 1\cdot 20 = 20$

b) $\det A = \det\begin{bmatrix}0 & -1 & 1 & 4\\ 3 & 2 & -2 & 1\\ 0 & 4 & 0 & 1\\ 1 & 0 & -1 & 1\end{bmatrix} = \det\begin{bmatrix}1 & 0 & -1 & 1\\ 0 & -1 & 1 & 4\\ 0 & 4 & 0 & 1\\ 3 & 2 & -2 & 1\end{bmatrix} = -\det\begin{bmatrix}1 & 0 & -1 & 1\\ 0 & 1 & -1 & -4\\ 0 & 4 & 0 & 1\\ 3 & 2 & -2 & 1\end{bmatrix}$

$= \det\begin{bmatrix}1 & 0 & -1 & 1\\ 0 & 1 & -1 & 4\\ 0 & 0 & 4 & 17\\ 3 & 2 & -2 & 1\end{bmatrix} = 4\det\begin{bmatrix}1 & 0 & -1 & 1\\ 0 & 1 & -1 & 4\\ 0 & 0 & 1 & 17/4\\ 0 & 2 & 1 & -2\end{bmatrix} = \frac{1}{4}\det\begin{bmatrix} & \text{etc} & \end{bmatrix}$

~~c)~~ $\det A = \det\begin{bmatrix}1 & 0 & 3\\ 8 & 1 & -1\\ 5 & 1 & -1\end{bmatrix} = \det\begin{bmatrix}1 & 0 & 3\\ 8 & 1 & -1\\ 3 & 0 & 0\end{bmatrix} = \det\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -25\\ 0 & 0 & -9\end{bmatrix}$

$= -\frac{1}{9}\det\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -25\\ 0 & 0 & 1\end{bmatrix} = 1$

c) $A = \begin{bmatrix}1 & 0 & 3\\ 8 & 1 & -1\\ 5 & 1 & -1\end{bmatrix} \Rightarrow \det\begin{bmatrix}1 & 0 & 3\\ 8 & 1 & -1\\ 5 & 1 & -1\end{bmatrix} = \det\begin{bmatrix}1 & 0 & 3\\ 8 & 1 & -1\\ 3 & 0 & 0\end{bmatrix} = \det\begin{bmatrix}1 & 0 & 3\\ 8 & 1 & -1\\ 0 & 0 & 9\end{bmatrix}$

$= \frac{81}{9}\det\begin{bmatrix}1 & 0 & 3\\ 8 & 1 & -1\\ 0 & 0 & 1\end{bmatrix} = \frac{81}{9}\det\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -25\\ 0 & 0 & 1\end{bmatrix} = \frac{81}{9} = 9$

~~c)~~ $\det A = \det\begin{bmatrix}1 & 3 & 2\\ -1 & -1 & -1\\ 2 & 3 & -2\end{bmatrix} = \det\begin{bmatrix}1 & 3 & 2\\ 0 & 2 & 1\\ 0 & 3 & 6\end{bmatrix} = \frac{1}{2}\det\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1/2\\ 0 & 3 & 6\end{bmatrix}$

$= \frac{1}{2}\det\begin{bmatrix}1 & 0 & 3/2\\ 0 & 1 & 1/2\\ 0 & 0 & 9/2\end{bmatrix} = \frac{1}{2}\cdot\frac{9}{2} = \frac{9}{4}$

d) $\det A = \det\begin{bmatrix}1 & 3 & 2\\ -1 & -1 & -1\\ 2 & 3 & -2\end{bmatrix} = \det\begin{bmatrix}1 & 3 & 2\\ 0 & 2 & 1\\ 0 & -3 & -6\end{bmatrix} = -2\det\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1/2\\ 0 & 3 & 6\end{bmatrix}$

$= -2\det\begin{bmatrix}1 & 3 & 2\\ 0 & 1 & 1/2\\ 0 & 0 & 9/2\end{bmatrix} = -2\cdot\frac{9}{2} = -9$

e) $\det A = \begin{bmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{bmatrix} = \det\begin{bmatrix}1 & 1 & 1\\ 0 & b-a & c-a\\ a & b^2-a^2 & c^2-a^2\end{bmatrix} = \det\begin{bmatrix}1 & 1 & 1\\ 0 & b-a & c-a\\ 0 & (b-a)(b+a) & (c-a)(c+a)\end{bmatrix}$

$= \det\begin{bmatrix}1 & 1 & 1\\ 0 & b-a & c-a\\ 0 & 0 & (c-a)(c+a)-(c-a)(b+a)\end{bmatrix} = \det\begin{bmatrix}1 & 1 & 1\\ 0 & b-a & c-a\\ 0 & 0 & (c-a)[c-b]\end{bmatrix}$

$= (b-a)(c-a)(c-b)$

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2) a) $A = \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -2x & 2 & 0 \\ x^2 & -2x & 1 \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} 2 & -2x & x^2 \\ 0 & 2 & -2x \\ 0 & 0 & 1 \end{bmatrix}$

$\text{adj } A \cdot A = \det A \cdot I = \begin{bmatrix} 2 & -2x & x^2 \\ 0 & 2 & -2x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow \det A = 2$

$\therefore A^{-1} = \begin{bmatrix} 1 & -x & \frac{x^2}{2} \\ 0 & 1 & x \\ 0 & 0 & 1/2 \end{bmatrix}$

c) $A = \begin{bmatrix} x^2 & \sin x & e^x \\ 2x & \cos x & e^x \\ 2 & -\sin x & e^x \end{bmatrix} = \begin{bmatrix} e^x(\sin x + \cos x) \\ \\ \\ \end{bmatrix}$

ARG!
(TOO HAIRY)

e) $A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \Rightarrow \begin{bmatrix} \cos x & -\sin x \\ +\sin x & \cos x \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$\text{adj } A \cdot A = \det A \cdot I = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A^{-1} = \text{adj } A$

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1) $$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ -1 & 1 & 1 & 3 \\ 1 & 9 & 3 & 0 \\ -3 & 5 & 2 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 10 & 4 & 3 \\ 0 & 8 & -4 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 9 & -7 \\ 0 & 0 & 0 & -12 \end{array}\right] \Rightarrow \text{NO}$$

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3) a) $$a_1\begin{bmatrix}1\\0\\0\end{bmatrix} + a_2\begin{bmatrix}0\\1\\0\end{bmatrix} + a_3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \text{INDEPENDENT}$$

b) $$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{DEPENDENT}$$

c) $$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\0 & 1 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 1 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \Rightarrow \text{IND.}$$

d) $$\begin{bmatrix}1 & 1 & 1\\2 & 3 & 4\\4 & 9 & 16\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 5 & 12\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\\0 & 0 & 1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \Rightarrow \text{IND.}$$

e) $$\begin{bmatrix}1 & 1 & 2\\-1 & 2 & 1\\2 & 1 & 3\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 2\\0 & 3 & 3\\0 & -1 & -1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix} \Rightarrow \text{DEP.}$$

f) $$\begin{bmatrix}1 & 1 & 1 & 0\\2 & 3 & 10 & -1\\3 & 5 & -5 & 17\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1 & 1 & 0\\0 & 1 & 8 & -1\\0 & 2 & -8 & 17\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -7 & 1\\0 & 1 & 8 & -1\\0 & 0 & -24 & 19\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -7 & 1\\0 & 1 & 0 & \\ & & & \end{bmatrix} \text{ARG!}$$

4) a) $A = \left[e^t \; 3e^{3t}\right]$

$A\alpha = \alpha_1 e^t + 3\alpha_2 e^{3t} = 0 \Rightarrow \alpha_1 + 3\alpha_2 e^{2t} = 0$

$\Rightarrow 6\alpha_2 e^{2t} = 0 \Rightarrow \alpha_1 = \alpha_2 = 0 \Rightarrow$ INDEP

b) $A\alpha = \alpha_1 2e^{2t} + \alpha_2 e^{2t} + 2\alpha_2 t e^{2t} = 0$

$\Rightarrow \alpha_1 2 + \alpha_2 + 2\alpha_2 t = 2\alpha_1 + \alpha_2(1+2t) = 0$

$\Rightarrow 2\alpha_2 = 0 \Rightarrow \alpha_2 = \alpha_1 = 0 \Rightarrow$ INDEP

c) $\alpha_1 \lambda e^{\lambda t} + \alpha_2 \mu e^{\mu t} = 0$

$\Rightarrow \alpha_1 \lambda + \alpha_2 \mu e^{(\mu-\lambda)t} = 0$

$\Rightarrow$ IND IF $\alpha_1 \neq \alpha_2$, DEP IF $\alpha_1 = \alpha_2$

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1) a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 5 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ⇒ FORMS BASIS

b) $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 6 & 1 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & +1 & +1 \\ 6 & 1 & 13 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ⇒ DOESN'T FORM BASIS

c) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 3 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ⇒ DOESN'T FORM BASIS

d) $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ⇒ DOESN'T FORM BASIS

e) NO!

f) YES

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4) a) $\begin{bmatrix} 2 & 0 & 0 \\ -1 & 7 & 0 \\ 5 & -2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 7 & 0 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ -1 & 7 & 0 & 0 \\ 5 & -2 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 7 & 0 & 1/2 \\ 0 & +2 & +1 & +5/2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/14 \\ 0 & 0 & 1 & 33/14 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} + \frac{1}{14}\begin{bmatrix} 0 \\ 7 \\ -2 \end{bmatrix} + \frac{33}{14}\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 5/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ -1/7 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -33/14 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow e_1 = \frac{1}{2} V_1 + \frac{1}{14} V_2 + \frac{33}{14} V_3$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ -1 & 7 & 0 & 1 \\ 5 & -2 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 1 \\ 0 & +2 & +1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/7 \\ 0 & 0 & 1 & -2/7 \end{array}\right]$$

$\Rightarrow e_2 = 1/7\, V_2 - 2/7\, V_3$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ -1 & 7 & 0 & 0 \\ 5 & -2 & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & +2 & +1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\Rightarrow e_3 = -V_3$

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$\vec{a} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{c} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{d} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{e} = \begin{bmatrix} 1 \\ -3 \\ 1 \\ 2 \end{bmatrix}$ $\vec{f} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

GIVEN BASIS $\vec{a}\vec{b}\vec{c} \in S$; $\vec{d}\vec{e}\vec{f} \in T$, FIND BASIS & DIMENSION OF $S \cap T$

$$\vec{w} = \alpha_1\vec{a} + \alpha_2\vec{b} + \alpha_3\vec{c} = B_1\vec{d} + B_2\vec{e} + B_3\vec{f}$$

$$\begin{bmatrix} 2 & 1 & 0 & 3 & 1 & 2 \\ 3 & 1 & 1 & 1 & -3 & -1 \\ 0 & 1 & -1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ -\beta_1 \\ -\beta_2 \\ -\beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & +1 & -1 & -4 & -2 \\ 0 & 1 & 1 & -2 & -9 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 & -4 & -2 \\ 0 & 0 & 0 & -2 & -2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 & -4 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -4 & -1 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$\alpha_1 - \beta_2 = 0$; $\alpha_2 + 4\beta_2 + \beta_3 = 0$; $\alpha_3 + 3\beta_2 + \beta_3 = 0$; $\beta_1 + \beta_2 + \beta_3 = 0$

$$\Rightarrow \vec{W} = \beta_2\vec{a} + (+4\beta_2 + \beta_3)\vec{b} + (+3\beta_2 + \beta_3)\vec{c}$$

$$\vec{W} = \beta_2(\vec{a} - 4\vec{b} - 3\vec{c}) + \beta_3(-\vec{b} - \vec{c}) = S \cap T$$

$$\vec{x} = \vec{a} - 4\vec{b} - 3\vec{c} = \begin{bmatrix} -2 \\ -4 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{Y} = -(\vec{b} + \vec{c}) = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$$

1) $$\begin{bmatrix} 4 & 2 & 0 & 2 \\ -1 & 3 & 7 & -11 \\ 2 & -1 & -4 & 7 \\ 1 & -2 & -5 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & -5 & 8 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 6 & -9 \\ 0 & 4 & 8 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 14 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -18 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 14 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{DIMENSION} = 3$$

$$V_1 = \begin{bmatrix} 4 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -2 \end{bmatrix} \quad V_3 = \begin{bmatrix} 2 \\ -11 \\ 7 \\ 8 \end{bmatrix} \qquad a = V_1;\ b = V_2;\ d = V_3$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ -1 & 3 & -11 & 7 \\ 2 & -1 & 7 & -4 \\ 1 & -2 & 8 & -5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 8 & -5 \\ 0 & 5 & -3 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 2 & -5 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 3 & -2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2/3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/3 \end{array}\right]$$

2) $$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 5 & 1 \\ 1 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -4\vec{a} + \vec{b} = \vec{c}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 1 & 5 & 3 \\ 1 & 4 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -2\vec{a} + \vec{b} = \vec{d}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 3 \\ 0 & 2 & 6 \\ 1 & 5 & 5 \\ 1 & 4 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & -10 \\ 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -10 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow -10\vec{a} + 3\vec{b} = \vec{e}$$

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$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$ FIND MIN. EQ

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & -2 & 4-\lambda \end{vmatrix} = 2-\lambda \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -2 & 4-\lambda \end{vmatrix} = [(2-\lambda)(1-\lambda)(4-\lambda) + 2(2-\lambda)](2-\lambda)$$

$$= [(2-\lambda)[(1-\lambda)(4-\lambda)+2](2-\lambda)$$

$$= (2-\lambda)^2[\lambda^2 - 5\lambda + 7] \qquad \lambda = \frac{5 \pm \sqrt{25-28}}{2}$$

$$\therefore (2I - A)^2 [A^2 - 5A + 7I] = \vec{0}$$

$$(2I - A)[A^2 - 5A + 7I] \overset{?}{=} \vec{0}$$

$$2I - A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 6 & -10 & 14 \end{bmatrix}$$

$$A^2 - 5A + 7I = \begin{bmatrix} 4 & 4 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & -10 & 14 \end{bmatrix} + \begin{bmatrix} -10 & -5 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 10 & -20 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(2I - A)(A^2 - 5A + 7I) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 6 & 2 & -2 \end{bmatrix}$$

$\therefore (2I - A)^2 (A^2 - 5A + 7A)$ IS MIN. EQ

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$$\ddot{\Phi} = \begin{bmatrix} -\left(\frac{K}{m}+\frac{g}{L}\right) & \frac{K}{m} & 0 \\ \frac{K}{m} & -\left(\frac{2K}{m}+\frac{g}{L}\right) & \frac{K}{m} \\ 0 & \frac{K}{m} & -\left(\frac{K}{m}+\frac{g}{L}\right) \end{bmatrix} \Phi$$

$$\left|\ddot{\Phi} - \lambda I\right| = \begin{vmatrix} -\left(\frac{K}{m}+\frac{g}{L}\right)-\lambda & \frac{K}{m} & 0 \\ \frac{K}{m} & -\left(\frac{2K}{m}+\frac{g}{L}\right)-\lambda & \frac{K}{m} \\ 0 & \frac{K}{m} & -\left(\frac{K}{m}+\frac{g}{L}\right)-\lambda \end{vmatrix}$$

$$= -\left[\frac{K}{m}+\frac{g}{L}+\lambda\right]\left[\frac{2K}{m}+\frac{g}{L}+\lambda\right]\left[\frac{K}{m}+\frac{g}{L}+\lambda\right] + \frac{2K^2}{m^2}\left(\frac{K}{m}+\frac{g}{L}+\lambda\right) = 0$$

$$= -\left[\frac{K}{m}+\frac{g}{L}+\lambda\right]^2\left[\frac{2K}{m}+\frac{g}{L}+\lambda\right] + \frac{2K^2}{m^2}\left(\frac{K}{m}+\frac{g}{L}+\lambda\right)$$

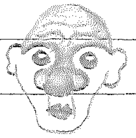
$$= \left(\frac{K}{m}+\frac{g}{L}+\lambda\right)\left[\left(\frac{2K}{m}+\frac{g}{L}+\lambda\right)\left(\frac{K}{m}+\frac{g}{L}+\lambda\right) - \frac{2K^2}{m^2}\right]$$

$$= -\left(\frac{K}{m}+\frac{g}{L}+\lambda\right)\left[\cancel{\frac{2K^2}{m^2}} + \frac{3K}{m}\left(\frac{g}{L}+\lambda\right) + \left(\frac{g}{L}+\lambda\right)^2 - \cancel{\frac{2K^2}{m^2}}\right]$$

$$0 = \left(\frac{K}{m}+\frac{g}{L}+\lambda\right)\left(\frac{g}{L}+\lambda\right)\left(\frac{g}{L}+\frac{3K}{m}+\lambda\right)$$

$$\lambda_1 = -\left(\frac{K}{m}+\frac{g}{L}\right);\ \lambda_2 = -\frac{g}{L} \quad \lambda_3 = -\left(\frac{g}{L}+\frac{3K}{m}\right)$$

$\lambda^3 - 2\lambda^2 + 3\lambda - 1 = 0$



$S_0 = -1$

$S_1 = 3\lambda - 1$

$S_2 = -2\lambda_2 + 3\lambda - 1$

$S_3 = A^3 - 2A^2 + 3A - I = 0$

$A^3 - 2A^2 + 3A - I = 0$

$S_1 = I + A + \frac{A^2}{2}$

$S_3 = \lambda^3 - 2\lambda^2 + 3\lambda - 1 = 0$

$\lambda^3 = 2\lambda^2 + 3\lambda + 1$

$P(\lambda) = \lambda^3 - 2\lambda^2 + 3\lambda - 1 = 0$

$\Rightarrow A^3 - 2A^2 + 3A - I = 0$

$A^3 = 2A^2 - 3A + I$

$S_1 = -I$

$S_2 = 3A - I$

Test Requirements

✓ To evaluate adj A " MATRIX OF COFACTORS OF A^T and to obtain A^{-1} from $A^{-1} = \frac{\text{adj}A}{\det A}$ if A is regular

✓ To determine whether a set of vectors is a vectorspace

✓ To " " " " " " " a subspace

✓ To " " " " " " is linearly dependent

✓ To find a basis if a generating set is given.

✓ To find the coordinates of a vector with respect to a given basis

✓ To find the dimension of a space spanned by a given generating set

✓ To find ~~the~~ a basis and the dimension of the intersection of two vector spac.

To know the relations between row rank, column rank of a matrix and the dimension of its solution space. Hence to find the ~~solution~~ dimension of the solution space of $A\bar{x} = \bar{o}$ without actually solving.

$A \in F_{mn}$, $rr(A) = r$ ⇒ SOLUTION W OF $AX=0$ IS A SUBSPACE OF F_{n1}, WITH DIMENSION $n-r$

3) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 0 & 3 & 2 & 2 \\ -1 & 0 & -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \Rightarrow \begin{array}{c|c} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} \alpha \\ \alpha_2 \\ -B_1 \\ -B_2 \end{matrix} \end{array}$

$W = \alpha_1 \vec{a} + \alpha_2 \vec{b} = B_1 \vec{c} + B_2 \vec{d}$

$\alpha_1 - B_2 = 0 \qquad -B_1 - B_2 = 0$

$\alpha_2 = 0$

$\Rightarrow w = B_2 \vec{a} \qquad w = B_1 \vec{c} - B_1 \vec{d} = -B_2 \vec{c} + B_2 \vec{d} = B_2 (\vec{c} - \vec{d})$

$\Rightarrow$ BASIS = $\vec{a}, (\vec{c} - \vec{d})$ FOR 2 DIMEN

Problems Linear Algebra

1 The vectorspace $S \subseteq R_4$ is spanned by the vectors $\vec{a} = (4,-1,2,1)^T$, $\vec{b} = (2,3,-1,-2)^T$, $\vec{c} = (0,7,-4,-5)^T$ and $\vec{d} = (2,-11,7,8)^T$
Find a basis for S and determine its dimension. Express $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ in terms of the basis.

2 Show that $\vec{a} = (0,0,1,1)^T$ and $\vec{b} = (1,2,5,4)^T$ are a basis for the space $S \subseteq R_4$ spanned by $\vec{c} = (1,2,1,0)^T$; $\vec{b} = (1,2,3,2)^T$, $\vec{e} = (3,6,5,2)$

3 The space $S_2 \subset R_4$ has a basis $\vec{a} = (1,1,0,-1)^T$, $\vec{b} = (1,2,3,0)^T$ and the space $T_2 \subset R_4$ has a basis $\vec{c} = (1,2,2,-2)^T$, $\vec{d} = (2,3,2,-3)^T$
Find a basis for $S \cap T$ and determine its dimension.

4 The space $S \subset R_4$ is spanned by $\vec{a} = (2,3,1,0)^T$, $\vec{b} = (1,2,0,0)^T$, $\vec{c} = (1,2,3,-1)^T$ and the space $T \subset R_4$ is spanned by $\vec{d}(3,0,2,1)^T$, $\vec{e} = (2,-1,1,0)^T$, $\vec{f} = (1,2,3,-1)^T$
Show that $\vec{a}$ $\vec{b}$ and $\vec{c}$ are a basis for S and that $\vec{d}$, $\vec{e}$ and $\vec{f}$ are a basis for T. Find a basis for $S \cap T$ and obtain its dimension.

5 The space $S \subset R_4$ is spanned by $\vec{a} = (0,1,2,3)^T$, $\vec{b} = (2,-1,0,-3)^T$. and the space $T \subset R_4$ is spanned by $\vec{c} = (1,-1,-1,5)^T$, $\vec{d} = (0,1,2,-5)^T$, $\vec{e} = (4,-1,2,5)^T$
Show that S and T have dimension 2. and obtain a basis for T.
Also obtain a basis for $S \cap T$ and determine its dimension.

Pg 216-7   2-16-71

1) a) $\begin{bmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{bmatrix} = (1+\lambda)^2+4 = \lambda^2+2\lambda+5 \Rightarrow \lambda = \frac{-2 \pm 4i}{2} = -1 \pm 2i$

FOR $\lambda_1 = -1+2i$

$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \Rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \Rightarrow U = \alpha \begin{bmatrix} 1 \\ +i \end{bmatrix}$

FOR $\lambda_2 = -1-2i$

$\begin{bmatrix} 2i & 2 \\ -2 & 2i \end{bmatrix} \Rightarrow \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \Rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \Rightarrow V = \beta \begin{bmatrix} 1 \\ -i \end{bmatrix}$

c) $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)-6 = 4-5\lambda+\lambda^2-6 = \lambda^2-5\lambda-2 \Rightarrow \lambda = \frac{5 \pm \sqrt{33}}{2}$

b) $\begin{vmatrix} -3-\lambda & 1 & 7 \\ 0 & 4-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (-3-\lambda)(4-\lambda)(2-\lambda) = 0 \Rightarrow \lambda_1 = -3;\ \lambda_2 = 4;\ \lambda_3 = 2$

FOR $\lambda_1 = -3$

$\begin{bmatrix} 0 & 1 & 7 \\ 0 & 7 & -1 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & -50 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow U_1 = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

FOR $\lambda_2 = 4$

$\begin{bmatrix} -7 & 1 & 7 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/7 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{7} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow U_2 = \alpha_2 \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$

FOR $\lambda_3 = 2$

$\begin{bmatrix} -5 & 1 & 7 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -10 & 0 & 15 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow U_3 = \alpha_3 \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \alpha_3' \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 2-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix} \Rightarrow (2-\lambda)(1-\lambda)+1 = \lambda^2 - 3\lambda + 3 \Rightarrow \lambda = \frac{3 \pm i\sqrt{3}}{2}$

FOR $\lambda_1 = \frac{3+i\sqrt{3}}{2}$

$\begin{bmatrix} \frac{+1-i\sqrt{3}}{2} & 1 \\ -1 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{+1-i\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow u_1 = \alpha_1' \begin{bmatrix} 1 \\ \frac{-1+i\sqrt{3}}{2} \end{bmatrix} = \alpha_1'' \begin{bmatrix} 2 \\ -1+i\sqrt{3} \end{bmatrix}$

FOR $\lambda = \frac{3-i\sqrt{3}}{2}$

$\begin{bmatrix} \frac{1+i\sqrt{3}}{2} & 1 \\ -1 & \frac{-1+i\sqrt{3}}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1+i\sqrt{3} & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow u_2 = \alpha_2 \begin{bmatrix} -2 \\ 1+i\sqrt{3} \end{bmatrix}$

k) $\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -6-\lambda \end{bmatrix} \Rightarrow \lambda^2(-6-\lambda)-6-11\lambda = -6\lambda^2-\lambda^3-6-11\lambda = -\lambda^3-6\lambda^2-11\lambda-6$

$= (\lambda+1)(\lambda+1)(\lambda-3)$

FOR $\lambda_1 = -1$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -6 & -11 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -11 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

FOR $\lambda = 3$

$\begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ -6 & -11 & -9 \end{bmatrix}$

54 B+

Linear Algebra Test 1 12/12/70

Name BOB MARKS Box 385-2

1 Given the matrices $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

which of the following exist: [✓ A+B], A+C, AB, BA, [✓ AC] [✓ CA]?

Evaluate whichever of these sums and products that exist.

$$A+B = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & -2 \\ 2 & 3 \end{bmatrix}$$ ✓

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$$AC = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 0 \\ -2 & 2 & 0 \\ 8 & 4 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 4 & 4 \end{bmatrix}$$

2 a) Define the inverse of a square matrix and find the inverse A^{-1} of $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

b) Write the system $\begin{matrix} 2x + y = 1 \\ 5x + 3y = 2 \end{matrix}$ in matrix notation

and solve by means of the inverse of the matrix of coefficients.

a) $AA^{-1} = A^{-1}A = I \ni I$ IS THE IDENTITY MATRIX

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ni A^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

$$\overset{6}{2}W + \overset{3}{Y} = \overset{3}{1} \qquad \overset{6}{2}X + \overset{3}{Z} = 0$$

$$\underline{5W + 3Y = 0} \qquad \underline{5X + 3Z = 1}$$

$$W = 3 \qquad X = -1$$

$$6 + Y = 1 \qquad -2 + Z = 0 \Rightarrow A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$ ✓

$$\Rightarrow Y = -5 \qquad \Rightarrow Z = 2$$

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b) $$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$A \; b = C$

$A^{-1}A\,b = A^{-1}C$

$b = A^{-1}C$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \cancel{\begin{bmatrix} 7 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \cancel{\begin{bmatrix} 5 \\ -8 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore X = 5; \; Y = -8$

3 Show that $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ satisfies the equation $A^2 - 5A + I = 0$

Derive that $A^{-1} = 5I - A$ and hence evaluate A^{-1};

$$A^2 = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 25 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 5 \\ 25 & 14 \end{bmatrix} + \begin{bmatrix} -10 & -5 \\ -25 & -15 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \quad \checkmark$$

10/10

$$A^2 - 5A + I = 0$$
$$A^{-1}(A \cdot A - 5A + I) = A^{-1}0 = 0$$
$$(A^{-1}A)A - A^{-1}A5 + A^{-1}I = 0$$
$$A - I5 + A^{-1} = 0 \Rightarrow A^{-1} = I5 - A$$

$$A^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \quad \checkmark$$

4 Solve the system of equations

$x_1 + x_2 + x_3 + x_4 = 0$

$x_1 + x_2 + x_3 - x_4 = 4$

$x_1 + x_2 - x_3 + x_4 = -4$

$x_1 - x_2 + x_3 + x_4 = 2$

by row reduction of the augmented matrix.

~~$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 1 & -1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_4 \to R_1 - R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 0 & 2 & 0 & 0 & -2 \end{array}\right] R_4 \to \tfrac{1}{2}R_4 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right]$~~

~~$R_3 \to R_3 - R_4 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right] R_2 \to R_2 - R_3 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right]$~~

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 1 & -1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_4 \to R_1 - R_4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 0 & 2 & 0 & 0 & -2 \end{array}\right] R_4 \to \tfrac{1}{2}R_4 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$R_2 \to R_2 - R_1 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} R_3 \to R_1 + R_3 \\ R_2 \to -\tfrac{1}{2}R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & +1 & -2 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right] R_3 \to \tfrac{1}{2}R_3 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\text{SWITCHING ROWS} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2 - R_3 - R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right] \Rightarrow$$

$x_4 = -2$

$x_3 = 2$

$x_2 = -1$

$x_1 = 1$

10/10 but clumsy

5 Show that the matrix $A = \begin{bmatrix}3&4\\1&2\end{bmatrix}$ is row equivalent to the unit matrix $I = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ and hence write A as the product of elementary matrices

I get $\begin{bmatrix}-3&-4\\4&5\end{bmatrix}$

6/10

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}3&4\\1&2\end{bmatrix}$$

$R_2 \to R_1$, $R_1 \to R_2$ $\quad E_{12}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}$

$R_2 \to 2R_1 - R_2$ — $\begin{bmatrix}0&1\\-1&2\end{bmatrix}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}1&2\\-1&0\end{bmatrix}$

$R_1 \to R_2$, $R_2 \to R_1$ — $\begin{bmatrix}-1&2\\0&1\end{bmatrix}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}-1&0\\1&2\end{bmatrix}$

$R_1 \to -R_1$ — $\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}1&0\\1&2\end{bmatrix}$

$R_2 \to R_2 - R_1$ $\begin{bmatrix}1&2\\-1&-1\end{bmatrix}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}1&0\\0&2\end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{bmatrix}1&2\\-\frac{1}{2}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

ELEMENTRY

E_2

$$\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\cdot\begin{bmatrix}1&0\\-1&1\end{bmatrix}\cdot\begin{bmatrix}-1&0\\0&1\end{bmatrix}\cdot$$

$$\cancel{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\cdot\begin{bmatrix}1&0\\2&-1\end{bmatrix}\cdot\cancel{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\begin{bmatrix}3&4\\1&2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

A $\quad I$

REDUCE TO I

not of types E, F, G

6 Reduce the matrices $\begin{bmatrix}1&2&3\\2&3&2\end{bmatrix}$ and $\begin{bmatrix}2&3&2\\1&1&-1\end{bmatrix}$ to REF and hence show that they are row equivalent.

~~$\begin{bmatrix}1&2&3\\2&3&2\end{bmatrix} \; R_2 \to 2R_1 - R_2 \; \begin{bmatrix}1&2&3\\0&1&4\end{bmatrix} \; R_1 \to R_1 - 2R_2 \; \begin{bmatrix}1&0&-5\\0&1&4\end{bmatrix}$~~

~~$\begin{bmatrix}2&3&2\\1&1&-1\end{bmatrix} \; \begin{matrix}R_1 \to R_2\\R_2 \to R_1\end{matrix} \; \begin{bmatrix}1&1&-1\\2&3&2\end{bmatrix} \; R_2 \to R_2 - 2R_1 \; \begin{bmatrix}1&1&-1\\0&1&0\end{bmatrix}$~~

R

$\begin{bmatrix}1&2&3\\2&3&2\end{bmatrix} \; R_1 \to 2R_1 - R_2 \; \begin{bmatrix}0&1&4\\2&3&2\end{bmatrix} \; \begin{matrix}R_1 \to R_2\\R_2 \to R_1\end{matrix} \; \begin{bmatrix}2&3&2\\0&1&4\end{bmatrix}$

$R_1 \to \frac{1}{2}R_1 \; \begin{bmatrix}1&3/2&1\\0&1&4\end{bmatrix} \; R_1 \to R_1 - \frac{3}{2}R_2 \; \begin{bmatrix}1&0&-5\\0&1&4\end{bmatrix} \Longleftarrow$

$\begin{bmatrix}2&3&2\\1&1&-1\end{bmatrix} \; \begin{matrix}R_1 \to R_2\\R_2 \to R_1\end{matrix} \; \begin{bmatrix}1&1&-1\\2&3&2\end{bmatrix} \; R_2 \to R_2 - 2R_1 \; \begin{bmatrix}1&1&-1\\0&1&4\end{bmatrix}$

$R_1 \to R_1 - R_2 \Rightarrow \begin{bmatrix}1&0&-5\\0&1&4\end{bmatrix} \Longleftarrow$

10/10

61 A

Linear Algebra Test 2 1/18/71

Name Bob Marks Box 385

1 Solve the set of equations that is characterized by the augmented matrix $\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & 6 & 2 \\ 3 & 2 & -4 & 3 & 9 & 3 \end{array}\right]$ and write your answer in the form of linear combination of columnvectors

5-2

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & 3 & -6 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_4 + 3x_5 = 1$$
$$x_2 - 2x_3 = 0$$

10/10

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2 Show that the set of equations $x - 4y + 5z = 1$, $2x - y + 3z = 2$, $3x + 2y + z = p$ is inconsistent for general values of p. Determine p such that the system can be solved.

$$\left[\begin{array}{ccc|c} 1 & -4 & 5 & 1 \\ 2 & -1 & 3 & 2 \\ 3 & 2 & 1 & p \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 5 & 1 \\ 0 & -7 & 7 & 0 \\ 0 & -14 & 14 & 3-p \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -4 & 5 & 1 \\ 0 & -7 & 7 & 0 \\ 0 & 0 & 0 & 3-p \end{array}\right]$$

CONSIST~~A~~eNT IF $P=3$

10/10

3 Find the row rank of the matrix

$$\begin{bmatrix} 3 & 1 & 2 & 4 \\ 5 & 2 & 3 & 6 \\ 4 & 1 & 3 & 6 \\ 5 & 1 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 4 & 1 & 3 & 6 \\ 5 & 1 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -1 & -2 \\ 4 & 1 & 3 & 6 \\ 5 & 1 & 4 & 8 \end{bmatrix}$$

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$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -1 & -2 \\ 0 & \cancel{1/3} & \cancel{-1/3} & \cancel{-2/3} \\ 5 & 1 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 4 & 8 \end{bmatrix}$$

$\frac{2}{3}+$

$\frac{4}{3}+\frac{2}{3}$

$\frac{20}{3}-\frac{24}{3}$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -1 & -2 \\ 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow RANK = 2$$

10/10

4 Find the invers of the matrix A by row reduction of the augmented matrix [A|I] where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & \cancel{0}3 & 3 & 0 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -5 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -21 & 3 & 3 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

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$$\Rightarrow A^{-1} = \begin{bmatrix} \overset{1}{\cancel{-21}} & -3 & \overset{2}{\cancel{3}} \\ -3 & \cancel{0}3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

5 Complete the following:

(i) If two rows of a determinant are interchanged, then its value.. CHANGES SIGN

(ii) If any row of a determinant is multiplied by p then its value. IS ~~DIVIDED~~ multiplied BY P

(iii) If k times a row is added to another row of a determinant then its value.. DOESN'T CHANGE

State the corresponding properties of columns.

Use the above properties to evaluate

① COLUMBS CHANGE ⇒ SIGN CHANGE

② COLUMN ~~ROW~~ • P ⇒ pDET ~~XP~~

③ K COLUMN X + COLUMN Y ⇒ NO CHANGE

$$A = \begin{vmatrix} x & 3x & 2x \\ e^x & e^x & e^x \\ 2 & 3 & -2 \end{vmatrix}$$

$$③ \rightarrow \det A = \begin{bmatrix} x & 2x & 2x \\ e^x & 0 & e^x \\ 2 & 1 & -2 \end{bmatrix} \xrightarrow{③} \begin{bmatrix} x & 2x & 0 \\ e^x & 0 & e^x \\ 2 & 1 & -3 \end{bmatrix}$$

$$\det A = xe^x \det \begin{bmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 2 & 3 & -2 \end{bmatrix}$$

$$\Rightarrow xe^x \det \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & -1 \\ 2 & 3 & -2 \end{bmatrix} \Rightarrow -2xe^x \det \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{2} \\ 2 & 3 & -2 \end{bmatrix}$$

$$\Rightarrow -2xe^x \det \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & -3 & -6 \end{bmatrix} \Rightarrow \overset{6}{4}xe^x \det \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1/2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \overset{6}{4}xe^x \det \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 3/2 \end{bmatrix}$$

$$\Rightarrow \det A = \overset{6}{4}xe^x \cdot \frac{3}{2} = \overset{9}{6}xe^x$$

8/10

6 Given the matrix $A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$, evaluate det A

(i) by expansion with respect to the elements in the first row

(ii) by expansion with respect to the elements in the second column.

Evaluate the cofactors of the elements a_{12} and a_{22}.

i) $\det A = 2(-1) - 2(-2) + (-2)(-1)$
$= -2 + 4 + 2 = 4$

ii) $\det A = (-1)(14) + 2(12) - 3(2)$
$= -14 + 24 - 6 = 4$

cofactor of $a_{12} = -[4-6] = 2$

" " $a_{22} = +[12] = 12$

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7 A skew-symmetric matrix is defined by the property $A^T = -A$

(i) Show that A is square and that $a_{ii} = 0$

(ii) If the order of a skew-symmetric matrix is odd prove that $\det A = 0$ (Hint: compare det A and det A^T)

i) a) LET A BE OF ORDER mn
$\Rightarrow \overset{nm}{A^T}$ OF ORDER nm AND $-A$ OF ORDER mn
$\therefore$ IF $A^T = -A$, BOTH MUST BE OF SAME ORDER
$\Rightarrow \overset{\text{ROW}}{n} = \overset{\text{ROW}}{m}$ AND $\overset{\text{COL.}}{m} = \overset{\text{COL}}{n}$ $\therefore$ BOTH $-A$ & A^T ARE OF ORDER mm OR nn, SQUARE MATRICES

b) ~~Let~~ $a_{ii} \in A \ni a_{ii} = k \Rightarrow \exists a_{nii} \in -A \ni \underset{n}{a_{ii}} = -k = -a_{ii}$
ALSO $\exists a_{Tii} \in A^T \ni a_{Tii} = k = a_{ii}$
$A^T = -A \Rightarrow a_{Tii} = \underset{\Uparrow}{a_{nii}} \Rightarrow a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

ii) FROM PREVIOUS SECTION, ~~$a_{ii} = 0$~~ $\Rightarrow a_{11} = 0$
FROM DEF OF DETERMINATES:

$\det A = \det A^T = \det(-A) = (-1)^n \det A$

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43 B

Linear Algebra and Matrices Test 3

Name Bob Marley Box 385

1 S is a subspace of R_4 spanned by the vectors $\vec{a} = (1,0,6,-5)^T$ $\vec{b} = (1,-2,4,1)^T$, $\vec{c} = (2,-3,9,-1)^T$ and $\vec{d} = (2,-5,7,5)^T$. Obtain a basis for S and find its dimension.

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -5 \\ 6 & 4 & 9 & 7 \\ -5 & 1 & -1 & 5 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -5 \\ 0 & -2 & -3 & -5 \\ 0 & 6 & 9 & 15 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

10

$\Rightarrow 2^{ND}$ DIMENSION

$$\begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 4 & 9 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{a} \,\&\, \vec{b}$$ ARE LINEARLY INDEPENDENT

$\therefore$ LET BASIS = $\vec{a}$ & $\vec{b}$

2 Find the coordinates of the vector $\vec{a} = (3,5,-2)^T$ relative to the basis $\vec{e}_1 = (1,1,1)^T$, $\vec{e}_2 = (0,2,3)^T$, $\vec{e}_3 = (0,2,-1)^T$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & -1 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & -1 & -5 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -4 & -8 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

9

$$\Rightarrow \vec{a} = 3\vec{e}_1 \mp \vec{e}_2 + 2\vec{e}_3$$

3. Which theorem enables you to conclude that the vectors $\vec{a}=(1,2,0)^T$, $\vec{b}=(1,-1,1)^T$, $\vec{c}=(2,1,1)^T$ and $\vec{d}=(3,1,2)^T$ are linearly dependent without any calculations? Find $\vec{a}$ is a linear combination of $\vec{b}$, $\vec{c}$ and $\vec{d}$. Can you also find $\vec{d}$ as a linear combination of $\vec{a}$, $\vec{b}$ and $\vec{c}$?

THERE ARE 4 VECTORS, ONLY THREE NEEDED TO DEFINE 3RD DIMENSION ⇒ ~~ONLY~~ THREE VECTORS AT THE MOST CAN BE IND.
(# DIM = # BASIS VECTORS, ALL OF WHICH ARE IND)

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 3 & 4 & 3 \\ 0 & 2 & 3 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & \frac{4}{3} & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

8

$\vec{a} = \vec{c} - \vec{b}$ ✓

NO. $\vec{a}, \vec{b}, \vec{c}$ ARE DEPENDENT   wrong reason.

4. Find the row rank and the column rank of the matrix $\underset{mn}{A} = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 5 & 1 & 4 \end{bmatrix}$ by obtaining a basis for the row space and also a basis for the column space. What is the dimension of $n-r=\text{DIM}$ ✓ the solution space of $A\vec{x}=\vec{0}$? and of $A^T\vec{y}=\vec{0}$?

$$\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 5 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & 2 \\ 5 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rr = 3   cr = 3   show

W IS SOL. OF $\overset{A\vec{x}=\vec{0}}{\Rightarrow}$ DIM W = n - rr = 3-3=0

V IS A SOL OF $A^T\vec{Y}=\vec{0} \Rightarrow$ DIM W = m - cr = 4-3 = 1

6

5 S and T are subspaces of R_4. The vectors $\vec{a} = (2,3,1,0)^T$, $\vec{b} = (1,2,0,0)^T$ and $\vec{c} = (0,0,3,-1)^T$ are a basis for S and the vectors $\vec{d} = (3,0,2,1)$, $\vec{e} = (0,1,2,-2)^T$ and $\vec{f} = (1,2,3,-1)^T$ are a basis for T. Obtain a basis for the intersection $S \cap T$ of S and T and find its dimension.

$$\begin{bmatrix} 2 & 1 & 0 & 3 & 0 & 1 \\ 3 & 2 & 0 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 2 & 3 \\ 0 & 0 & -1 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ -\beta_1 \\ -\beta_2 \\ -\beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 & 2 & 3 \\ 2 & 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 & 2 & 2 & 3 \\ 0 & 1 & -6 & -1 & -4 & -5 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 2 & -9 & -6 & -5 & -7 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 & -4 & 0 \\ 0 & 1 & 0 & -7 & 8 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 3 & -4 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 & -4 & 0 \\ 0 & 1 & 0 & -7 & 8 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & +1 & +3 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -19 & 0 \\ 0 & 1 & 0 & 0 & 29 & 1 \\ 0 & 0 & 1 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{bmatrix}$$

$W = \alpha_1 \vec{a} + \alpha_2 \vec{b} + \alpha_3 \vec{c} = \beta_1 \vec{d} + \beta_2 \vec{e} + \beta_3 \vec{f}$

$\alpha_1 + 19\beta_2 = 0$; $\alpha_2 - 29\beta_2 + \beta_3 = 0$

$\alpha_3 - 5\beta_2 - \beta_3 = 0$ $\quad -\beta_1 - \beta_2 = 0$

$w = -19\beta_2 \vec{a} + (29\beta_2 + \beta_3)\vec{c} + (5\beta_2 + \beta_3)\vec{d}$

$= \beta_2(-19\vec{a} + 29\vec{c} + 5\vec{d}) + \beta_3(\vec{c} + \vec{d})$

BASIS FOR $S \cap T$ (SECOND DIMENSION):

$V_1 = -19\vec{a} + 29\vec{c} + 5\vec{d}$

$V_2 = \vec{c} + \vec{d}$

10

well done

SOLUTIONS

[illegible]

[illegible]

[illegible] basis for [illegible] its dimension

Solution: the problem is to check whether $\vec{a}$ $\vec{b}$ $\vec{c}$ and $\vec{d}$ are linearly dependent, that is to solve $\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}+\delta\vec{d}=\vec{0}$ for $\alpha, \beta, \gamma, \delta$.

$$\begin{matrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -5 \\ 6 & 4 & 9 & 7 \\ -5 & 1 & -1 & 5 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & -2 & -3 & -5 \\ 0 & 6 & 9 & 15 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 0 & .5 & -.5 \\ 0 & 1 & 1.5 & 2.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

or $\alpha = -\frac{1}{2}\gamma+\frac{1}{2}\delta \quad \beta = -\frac{3}{2}\gamma-\frac{5}{2}\delta$ or $(-\frac{1}{2}\gamma+\frac{1}{2}\delta)\vec{a}+(-\frac{3}{2}\gamma-\frac{5}{2}\delta)\vec{b}+\gamma\vec{c}+\delta\vec{d}=\vec{0}$

or $\gamma(\vec{c}-\frac{1}{2}\vec{a}+\frac{3}{2}\vec{b})+\delta(\vec{d}+\frac{1}{2}\vec{a}-\frac{5}{2}\vec{b})=\vec{0}$ that is $\vec{c}=\frac{1}{2}\vec{a}-\frac{3}{2}\vec{b}$, $\vec{d}=-\frac{1}{2}\vec{a}+\frac{5}{2}\vec{b}$

Find the coordinates of the vector $\vec{a}$ [illegible]

the basis $\vec{e}_1$ [illegible] $\vec{e}_3$ [illegible]

Solution we want $\alpha\ \beta\ \gamma$ in $\alpha\vec{e}_1+\beta\vec{e}_2+\gamma\vec{e}_3=\vec{a}$

$$\left.\begin{matrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 1 & 3 & -1 \end{matrix}\right| \begin{matrix} 3 \\ 5 \\ -2 \end{matrix} \overset{R}{\sim} \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 3 & -1 \end{matrix}\right| \begin{matrix} 3 \\ 2 \\ -5 \end{matrix} \overset{R}{\sim} \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{matrix}\right| \begin{matrix} 3 \\ 1 \\ -8 \end{matrix} \overset{R}{\sim} \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} 3 \\ 1 \\ 2 \end{matrix} \overset{R}{\sim} \left.\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right| \begin{matrix} 3 \\ -1 \\ 2 \end{matrix}$$

Hence $\vec{a} = 3\vec{e}_1-\vec{e}_2+2\vec{e}_3$

Solution Theorem: A set of k vectors in an n dimensional space is linearly dependent if $k \geq n$. Here $k = 4$, $n \leq 3$

Try again $\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} + \delta\vec{d} = 0$ or

$$\begin{matrix} 1 & 1 & 2 & 3 \\ 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 1 & 2 & 3 \\ 0 & -3 & -3 & -5 \\ 0 & 1 & 1 & 2 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{matrix} \sim \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \text{ or } \begin{matrix} \alpha = -\gamma \\ \beta = -\gamma \\ \delta = 0 \end{matrix}$$

Hence $-\gamma\vec{a} - \gamma\vec{b} + \gamma\vec{c} = -\gamma(\vec{a} + \vec{b} - \vec{c}) = \vec{0}$ or $\vec{a} = \vec{c} - \vec{b}$

Since $\delta = 0$ $\vec{d}$ cannot be expressed in terms of the other vectors

Solution

By inspection column1 = column2 + column3. Columns 2 and 3 indepent and are basis for columnspace. The dimension hence is 2. By inspection $row_1 = 2row_2 + row_3$ and $row_4 = 4row_2 + row_3$. Row 2 and row 3 are independent and are a basis for row space, which hence has dimension 2.

In $A\vec{x} = \vec{0}$ $\vec{x} \in \mathcal{F}_{3,1}$ $\rightarrow$ dim solution space $3 - 2 = 1$

In $A^T\vec{y} = \vec{0}$ $\vec{y} \in \mathcal{F}_{4,1}$ $\rightarrow$ dim solution space $4 - 2 = 2$.

[illegible] the [illegible] of R_4 [illegible]

$\vec b$ = [illegible] and $\vec c$ = [illegible] are a basis for S and the

vectors $\vec d$ = [illegible], $\vec e$ = (0, 1, 2, [illegible]) and $\vec f$ = [illegible]

a basis for T. Obtain a basis for the intersection [illegible]

of S and T and find its dimension

For any vector $\vec w \in S \cap T$: $\vec w = \alpha_1 \vec a + \alpha_2 \vec b + \alpha_3 \vec c = \beta_1 \vec d + \beta_2 \vec e + \beta_3 \vec f$ (1)

First try to determine α's + β's from $\alpha_1 \vec a + \alpha_2 \vec b + \alpha_3 \vec c - \beta_1 \vec d - \beta_2 \vec e - \beta_3 \vec f = \vec 0$

$$\begin{matrix} 2 & 1 & 0 & 3 & 0 & 1 \\ 3 & 2 & 0 & 0 & 1 & 2 \\ 1 & 0 & 3 & 2 & 2 & 3 \\ 0 & 0 & -1 & 1 & -2 & -1 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 0 & 3 & 2 & 2 & 3 \\ 0 & 1 & -6 & -1 & -4 & -5 \\ 0 & 2 & -9 & -6 & -5 & -7 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 0 & 3 & 2 & 2 & 3 \\ 0 & 1 & -6 & -1 & -4 & -5 \\ 0 & 0 & 3 & -4 & 3 & 3 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{matrix} \overset{R}{\sim}$$

$$\begin{matrix} 1 & 0 & 3 & 2 & 2 & 3 \\ 0 & 1 & -6 & -1 & -4 & -5 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 0 & 0 & 5 & -4 & 0 \\ 0 & 1 & 0 & -7 & 8 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{matrix} \overset{R}{\sim} \begin{matrix} 1 & 0 & 0 & 0 & -19 & 0 \\ 0 & 1 & 0 & 0 & 29 & 1 \\ 0 & 0 & 1 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{matrix}$$

Hence $\alpha_1 = -19\beta_2$, $\alpha_2 = 29\beta_2$, $\alpha_3 = 5\beta_2 + \beta_3$, $\beta_1 = -3\beta_2$

Substitution in (1) yields

$$\vec w = -19\beta_2 \vec a + 29\beta_2 \vec b + \beta_3 \vec b + 5\beta_2 \vec c + \beta_3 \vec c = -3\beta_2 \vec d + \beta_2 \vec e + \beta_3 \vec f$$

$$= -\beta_2 (19\vec a - 29\vec b - 5\vec c) + \beta_2 (\vec b + \vec c) = -\beta_2 (3\vec d - \vec e) + \beta_3 \vec f$$

This is a 2-dimensional space with basis

$$\vec u = 19\vec a - 29\vec b - 5\vec c = 3\vec d - \vec e = \begin{bmatrix} 9 \\ -1 \\ 4 \\ 5 \end{bmatrix} \text{ and } \vec v = \vec b + \vec c = \vec f = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

SATISFACTORY

Name: BOB MARKS 385-2

Find the characteristic equation, eigenvalues and eigenvectors of

$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$ Obtain its inverse by aid of the Cayley-Hamilton equation.

$$\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda)^2(5-\lambda) \Rightarrow 0$$

CHAR. EQ

$$P(\lambda)=(2-\lambda)^2(5-\lambda)$$

$$\Rightarrow \text{EIGEN VALUES} = (2,5)$$

FOR $\lambda = 2$

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{vmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} Y+2Z=0 \Rightarrow Y=0 \\ 3Z=0 \Rightarrow Z=0 \end{matrix} \quad X_1 = \alpha_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ ✓

FOR $\lambda = 5$

$$\begin{vmatrix} -3 & 1 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{vmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -3X+Y+2Z=0 \\ Y=Z \\ -3X+3Z=0 \Rightarrow X=Z \end{matrix} \Rightarrow X_2 = \alpha_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ ✓

(CONT ON BACK)

Find the characteristic equation, eigenvalues and eigenvectors of

$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}$ Obtain its minimum equation.

$4+12-$

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 1 & -2 \\ 1 & 1-\lambda & -2 \\ -2 & -2 & 4-\lambda \end{vmatrix} = (1-\lambda)^2(4-\lambda)+4+4-4(1-\lambda)-(4-\lambda)-4(1-\lambda)$$

$$= (1-\lambda)^2(4-\lambda)+\cancel{8}+\cancel{8}(\cancel{1}-\lambda)-(4-\lambda) = (1-\lambda)^2(4-\lambda)-$$

~~$= (1-\lambda)[(1-\lambda)(4-\lambda)-8]+$~~

~~$= (1-\lambda)^2(4-\lambda)-2($~~

~~$= (1-\lambda)^2(4-\lambda)+8[\lambda]-(4-\lambda)$~~

~~$= (4-\lambda)[(1-\lambda)^2-1]+8\lambda$~~

BY INSPECTION, EIGEN ~~VALUES~~ ONE = VECTOR $= c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda=0$

(BACK)

The matrix $A =$ [illegible] has the eigenvalues $-3, 2, 4$.

Determine a similarity transformation P such that $P^{-1}AP$ is a diagonal matrix. Invert P and evaluate $P^{-1}AP = D$ and verify that it is a diagonal matrix with the eigenvalues on the main diagonal.

$$\begin{bmatrix} 0 & -2 & 2 \\ & & \\ & & \end{bmatrix}$$

MATRICES and LINEAR DIFFERENTIAL EQUATIONS

1 Introduction The purpose of these notes is to develop methods for solving the equation $\dot{x}(t) = Ax(t) + f(t)$ where A is a square matrix with constant elements and $x(t)$ and $f(t)$ are column vectors. Sections 2 and 3 contain some required matrix theory, section 4 solves the homogeneous equation and in sections 5 and 6 the non-homogeneous equation is solved by means of the method of variation of parameters and the method of undetermined constants respectively.

2 The Cayley-Hamilton Theorem Let $\det(A - \lambda I) = P(\lambda) = a_0 + a_1\lambda + \cdots$ with $a_n = (-1)^n$ be the characteristic polynomial of the square matrix

We then have the famous Cayley Hamilton theorem

Theorem A matrix satisfies its own characteristic equation or

$$P(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$$

Proof: We know that $(A - \lambda I)\,\text{adj}(A - \lambda I) = I \det(A - \lambda I) = P(\lambda)$

Since $P(\lambda)$ is a polynomial of degree n and since $A - \lambda I$ is of degree 1, $\text{adj}(A - \lambda I)$ must be of degree $n-1$ in

Moreover $\text{adj}(A - \lambda I)$ is a matrix and if we write it as a polynomial in λ, the coefficients must be matrices

We now put

$$\text{adj}(A - \lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}$$

Hence we have

$$(A - \lambda I)\,\text{adj}(A - \lambda I) = AB_0 + (AB_1 - B_0)\lambda + \cdots + (AB_{n-1} - B_{n-2})\lambda^{n-1} -$$

$$= P(\lambda)I = a_0 I + a_1 \lambda I + \cdots + a_n \lambda^n I$$

Equating coefficients we find

$AB_0 = a_0 I$ I $AB_0 = a_0 I$

$AB_1 - B_0 = a_1 I$ A $A^2B_1 - AB_0 = a_1 A$

$AB_2 - B_1 = a_2 I$ A^2 $A^3B_2 - A^2B_1 = a_2 A^2$

— — — — — — — — — — — — — — —

$AB_{n-2} - B_{n-3} = a_{n-2} I$ A^{n-2} $A^{n-1}B_{n-2} - A^{n-2}B_{n-3} = a_{n-2} A^{n-2}$

$AB_{n-1} - B_{n-2} = a_{n-1} I$ A^{n-1} $A^nB_{n-1} - A^{n-1}B_{n-2} = a_{n-1} A^{n-1}$

$-B_{n-1} = a_n I$ A^n $-A^nB_{n-1} = a_n A^n$

Adding the equation in the second column yields the theorem

If the zeros of the characteristic ~~equation~~ polynomial are $\lambda_1, \lambda_2, \ldots \lambda_n$ it can be written in the form

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where some roots may be equal. The Cayley-Hamilton theorem then takes the form

$$P(A) = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I) = O.$$

In the case of multiple eigenvalues repeated factors will occur in Cayley-Hamilton equation $P(A) = O$. In that situation it sometimes happens that the matrix satisfies a polynomial of degree less than n that can be obtained by lowering the exponent of the repeated factor to at most 1. The equation of lowest degree that is satisfied by A is called the minimum equation and will be denoted by $M(A) = O$. The corresponding polynomial $M(\lambda)$ is called the reduced characteristic equation. Although more sophisticated method exist it is for small matrices simplest to determine $M(A)$ by substitution of A into the possible alternatives

Examples

1 The matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 8 & 0 & 2 \\ 0 & -1 & 5 \end{bmatrix}$ has characteristic ~~equation~~ polynom.

$P(\lambda) = -8 - 2\lambda + 5\lambda^2 - \lambda^3 = -(\lambda+1)(\lambda-2)(\lambda-4)$. Hence A satisfies $A^3 - 5A^2 + 2A + 8I = O$ and cannot satisfy an equation of lower degree since all eigenvalues are different.

2 The matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ 10 & -4 & 5 \\ 5 & -4 & 6 \end{bmatrix}$ has characteristic polynomial

$P(\lambda) = 2 - 5\lambda + 4\lambda^2 - \lambda^3 = -(\lambda-1)^2(\lambda-2)$. Hence A certainly satisfies $A^3 - 4A^2 + 5A - 2I = O$ and might satisfy $(A-I)(A-2I) = A^2 - 3A + 2I = O$ Substitution shows however that the latter equation is not satisfied.

3 The matrix $A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$ has the characteristic polynomial

$P(\lambda) = 108 - 81\lambda + 18\lambda^2 - \lambda^3 = -(\lambda-3)^2(\lambda-12) = 0$ Hence A certainly satisfies $A^3 - 18A^2 + 81A - 108I = O$ and might satisfy $(A-3I)(A-12I) = O$ or $A^2 - 15A + 36I = O$. Substitution shows that the latter equation indeed is satisfied and that hence is the minimum equation.

4 The matrix $C = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$ has the characteristic polynomial

$P(\lambda) = (\lambda-2)^3(\lambda-3)$. Hence A certainly satisfies $(A-2I)^3(A-3I) = O$ and might satisfy $(A-2I)^2(A-3I) = O$ or even $(A-2I)(A-3I) = O$ Substitution shows that the second equation is satisfied but the third not. The minimum equation for A hence is $(A-2I)^2(A-3I) = O$ or $A^3 - 7A^2 + 16A - 12I = O$

The characteristic and minimum equations provide another method of determining the inverse of a matrix. If A satisfies the Cayley-Hamilton equation $a_0 I + a_1 A + a_2 A^2 + \dots + a_n A^n = 0$ then, multiplying by A^{-1}, we find

$$A^{-1} = -\frac{1}{a_0}(a_1 I + a_2 A + \dots + a_n A^{n-1})$$

Using the minimum equation $M(A) = m_0 I + m_1 A + \dots + m_q A^q$ instead we find

$$A^{-1} = -\frac{1}{m_0}(m_1 I + m_2 A + \dots + a_q A^{q-1})$$

We moreover have the relation for regular and singular matrices

$$\operatorname{adj} A = -(a_1 I + a_2 A + \dots + a_n A^{n-1})$$

This last one can be proved by means of the relations used in the proof of the Cayley-Hamilton theorem. Since $\det(A - \lambda I) = a_0 + a_1 \lambda + \dots + a_n \lambda^n$ we have, taking $\lambda = 0$: $\det A = a_0$. Hence $a_0 = 0$ for a singular matrix. From $\operatorname{adj}(A - \lambda I) = B_0 + B_1 \lambda + \dots + B_{n-1} \lambda^{n-1}$ it follows that $\operatorname{adj} A = B_0$. Finally $-B_0 = AB_1 - B_0 + A(AB_2 - B_1) + \dots + A^{n-2}(AB_{n-1} - B_{n-2}) - A^{n-1}B_{n-1}$

$$= a_1 I + a_2 A + \dots + a_{n-1} A^{n-2} + a_n A^{n-1}$$

A further consequence of the Cayley-Hamilton is that all polynomials in A can be reduced to polynomials of at most degree n-1. Functions of square matrices can be defined by means of a power series in that matrix, for example $e^A = I + A + \frac{A^2}{2} + \dots + \frac{A^n}{n!} + \dots$. As in the case of series of constants the sum of the infinite series of matrices is defined as the limit of the sequence of partial sums. In the case of a matrix however, each partial sum can be reduced to a polynomial of degree n-1, where n is the order of the matrix, or lower if the Cayley Hamilton equation is not the minimum equation. In the limit we obtain the surprising result that any matrix function, defined by a power series, is equivalent to a matrix polynomial of at most degree n-1. In the next section a method will be developed for finding that polynomial

3 The Lagrange - Sylvester Theorem

We now proceed to express a polynomial in A of degree m or greater as a polynomial of degree m-1 where m is the degree of the reduced characteristic polynomial (that is the degree of the minimum equation). We remember that the reduced characteristic polynomial has the same eigenvalues as the characteristic polynomial, although possibly with a different multiplicity. Now let

$$F(A) = q_0 I + q_1 A + \cdots + q_r A^r \qquad r \geq m$$

be a matrix polynomial. In principle the minimum equation $M(A) = 0$ can be used to reduce it to a polynomial of degree m-1 in A, say

$$F(A) = c_0 I + c_1 A + \cdots + c_{m-1} A^{m-1}$$

Let $\lambda_1, \lambda_2 \ldots \lambda_m$ be the eigenvalues of A and suppose that the reduced characteristic equation $M(\lambda) = 0$ has no multiple eigenvalues. Since each root of the characteristic equation is also a root of the reduced characteristic equation we have that $M(\lambda_s) = 0$ for each eigenvalue λ_s. Considering the polynomial

$$F(\lambda_s) = q_0 + q_1 \lambda_s + \cdots + q_r A^r$$

it can be reduced to a polynomial of degree m-1 in exactly the same way as the matrix polynomial F(A):

$$F(\lambda_s) = c_0 + c_1 \lambda_s + \cdots + c_{m-1} \lambda_s^{m-1}$$

We now have the m+1 relations

$$F(A) = c_{m-1} A^{m-1} + \cdots + c_1 A + c_0 I$$

$$F(\lambda_1) = c_{m-1} \lambda_1^{m-1} + \cdots + c_1 \lambda_1 + c_0$$

$$- - - - - - - - - - - -$$

$$F(\lambda_m) = c_{m-1} \lambda_m^{m-1} + \cdots + c_1 \lambda_m + c_0$$

We now eliminate the unknown coefficients c_k from this system. The result can be written:

$$\begin{vmatrix} F(A) & A^{m-1} & \cdots & A & I \\ F(\lambda_1) & \lambda_1^{m-1} & \cdots & \lambda_1 & 1 \\ - & - & - & - & - \\ F(\lambda_m) & \lambda_m^{m-1} & \cdots & \lambda_m & 1 \end{vmatrix} = 0$$

Expanding this determinant with respect to the elements in the first column we obtain

$$D_0\, F(A) = D_1\, F(\lambda_1) + D_2\, F(\lambda_2) + \cdots + D_m\, F(\lambda_m)$$

where

$$D_0 = \begin{vmatrix} \lambda_1^{m-1} & - & - & \lambda_1 & 1 \\ \lambda_2^{m-1} & - & - & \lambda_2 & 1 \\ - & - & - & - & - \\ \lambda_m^{m-1} & - & - & \lambda_m & 1 \end{vmatrix}$$

and D_k is obtained from D_0 by replacing λ_k^r by A^r throughout. For example

$$D_1 = \begin{vmatrix} A^{m-1} & \cdots & A & I \\ \lambda_2^{m-1} & \cdots & \lambda_2 & 1 \\ - & - & - & - \\ \lambda_m^{m-1} & \cdots & \lambda_m & 1 \end{vmatrix}$$

All determinants D are Vandermonde determinants and are known to equal

$$D_0 = (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_m)(\lambda_2-\lambda_3)\cdots(\lambda_1-\lambda_m)\cdots(\lambda_{m-1}-\lambda_m)$$

$$D_1 = (A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_m I)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_m)\cdots(\lambda_{m-1}-\lambda_m)$$

and similarly for D_2, D_3 etc.

We now introduce Lagrangian coefficients, defined by $L_1(A) = D_1/D_0$, $L_2(A) = D_2/D_0$ and so forth.

Cancelling common factors we find

$$L_1(A) = \frac{(A-\lambda_2 I)(A-\lambda_3 I)\cdots(A-\lambda_m I)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_m)}$$

$$L_2(A) = \frac{(A-\lambda_1 I)(A-\lambda_3 I)\cdots(A-\lambda_m I)}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_m)}$$

and so on. We now have shown that

$$F(A) = L_1(A) F(\lambda_1) + L_2(A) F(\lambda_2) + \cdots + L_m(A) F(\lambda_m)$$

The same formula holds for a matrix function since it is the limit of a sequence of polynomials. We now have proved

Lagrange's Theorem

Let the matrix A have minimum equation $M(A) = 0$ with m simple eigenvalues. Then a function $F(A)$ can be written

$$F(A) = \sum_{k=1}^{m} F(\lambda_k) L_k(A)$$

It is seen that the coefficients $L_k(A)$ are polynomials of degree $m-1$ in A and that they only depend on A and its eigenvalues and not on the particular function $F(A)$. By taking $F(A) = A^0 = I$ we find the special relation

$$I = L_1(A) + L_2(A) + \cdots + L_m(A).$$

Taking $F(A) = L_1^2(A)$ we find

$$L_1^2(A) = L_1^2(\lambda_1) L_1(A) + L_1^2(\lambda_2) L_2(A) + \cdots + L_1^2(\lambda_m) L_m(A)$$

It is easily seen that $L_1(\lambda_1) = 1$ and $L_1(\lambda_2) = L_1(\lambda_3) = \cdots = L_1(\lambda_m) = 0$ so that $L_1^2(A) = L_1(A)$. In general it can be shown that $L_m^2(A) = L_m(A)$. Taking $F(A) = L_1(A) L_2(A)$ we find

$$L_1(A) L_2(A) = L_1(\lambda_1) L_2(\lambda_1) L_1(A) + L_1(\lambda_2) L_2(\lambda_2) L_2(A) + \cdots = 0$$

and in general $L_m(A) L_n(A) = 0$ for $m \neq n$.

Examples

1 Express e^{At} as a polynomial in A if $A = \begin{bmatrix} 2 & -3 & 3 \\ 4 & -5 & 3 \\ 4 & -4 & 2 \end{bmatrix}$

The eigenvalues are $-2, -1, 2$. They are all different and characteristic equation cannot be reduced. Using Lagrange's Theorem we evaluate

$$e^{At} = \frac{(A+I)(A-2I)}{(-2+1)(-2-2)} e^{-2t} + \frac{(A+2I)(A-2I)}{(-1+2)(-1-2)} e^{-t} + \frac{(A+2I)(A+I)}{(2+2)(2+1)} e^{2t}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} e^{2t}$$

2 Express e^{At} as a polynomial in A if $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

The eigenvalues are $1, 1, 4$. Since 1 is a double eigenvalue we try whether $(A-I)(A-4I) = 0$ is satisfied. It is and that shows that it is the minimum equation. Hence

$$e^{At} = \frac{A-4I}{1-4} e^{t} + \frac{A-I}{4-1} e^{4t}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} e^{t} + \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} e^{4t}$$

3 Express e^{At} as a polynomial in A if $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 0 & 2 \\ -2 & 1 & -1 \end{bmatrix}$

The eigenvalues are $1, i, -i$. Proceeding as before

$$e^{At} = \frac{(A-iI)(A+iI)}{(1-i)(1+i)} e^{t} + \frac{(A-I)(A+iI)}{(i-1)(i+i)} e^{it} + \frac{(A-I)(A-iI)}{(-i-1)(-i-i)} e^{-it}$$

Performing some algebra of complex numbers we find

$$e^{At} = \frac{1}{2}(A^2+I)e^{t} - \frac{1}{4}(A-I)\Big[\{A-I+i(A+I)\}e^{it} + \{A-I-i(A+I)\}e^{-it}\Big]$$

$$= \frac{1}{2}(A^2+I)e^{t} - \frac{1}{4}(A-I)^2(e^{it}+e^{-it}) - \frac{i}{4}(A^2-I)(e^{it}-e^{-it})$$

$$= \frac{1}{2}(A^2+I)e^{t} - \frac{1}{2}(A-I)^2\cos t + \frac{1}{2}(A^2-I)\sin t$$

Finally substituting for A

$$e^{At} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ -2 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} e^{t} + \frac{1}{2}\begin{bmatrix} -4 & 2 & -4 \\ -4 & 2 & -4 \\ 3 & -1 & 2 \end{bmatrix} \cos t + \frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & 1 & -2 \end{bmatrix} \sin t$$

If the reduced characteristic equation contains multiple roots Lagrange's theorem has to be generalized. In its generalized form it is known as the Lagrange-Sylvester Theorem. We will not derive it but rather show in an example how a suitable formula can be found in each particular instance.

Let A be a third order matrix with one double and one simple eigenvalue. We want to find a polynomial expression for F(A). To obtain it we consider a matrix with three distinct simple eigenvalues λ_1, λ_2 and λ_3 and take the limit for $\lambda_2 \rightarrow \lambda_1$.

We have already seen that F(A) satisfies the determinantal equation

$$\begin{vmatrix} F(A) & A^2 & A & I \\ F(\lambda_1) & \lambda_1^2 & \lambda_1 & 1 \\ F(\lambda_2) & \lambda_2^2 & \lambda_2 & 1 \\ F(\lambda_3) & \lambda_3^2 & \lambda_3 & 1 \end{vmatrix} = (\lambda_1 - \lambda_2) \begin{vmatrix} F(A) & A^2 & A & I \\ F(\lambda_1) & \lambda_1^2 & \lambda_1 & 1 \\ \frac{F(\lambda_1) - F(\lambda_2)}{\lambda_1 - \lambda_2} & \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 - \lambda_2} & \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} & 0 \\ F(\lambda_3) & \lambda_3^2 & \lambda_3 & 1 \end{vmatrix} = 0$$

by the rules of determinant manipulation. First dropping the factor $\lambda_1 - \lambda_2$ and then taking the limit for $\lambda_2 \rightarrow \lambda_1$ we find

$$\begin{vmatrix} F(A) & A^2 & A & I \\ F(\lambda_1) & \lambda_1^2 & \lambda_1 & 1 \\ F'(\lambda_1) & 2\lambda_1 & 1 & 0 \\ F(\lambda_3) & \lambda_3^2 & \lambda_3 & 1 \end{vmatrix} = 0$$

What happened is that in the original determinant the row $(F(\lambda_2), \lambda_2^2, \lambda_2, 1)$ has been replaced by the derivative of the [illegible]

Expansion of the determinant with respect to the elements in the first column yields after som algebra

$$F(A) = L_1^{(0)}(A)\,F(\lambda_1) + L_1^{(1)}(A)\,F'(\lambda_1) + L_3(A)\,F(\lambda_3)$$

where $L_1^{(0)}(A) = I - \frac{(A-\lambda_1 I)^2}{(\lambda_1-\lambda_3)^2}$, $L_1^{(1)}(A) = \frac{(A-\lambda_1 I)(A-\lambda_3 I)}{\lambda_1-\lambda_3}$, $L_3(A) = \frac{(A-\lambda_1 I)^2}{(\lambda_3-\lambda_1)^2}$

For a matrix with one, triple, eigenvalue in the reduced characteristic equation we derive in a similar way the determinantal equation

$$\begin{vmatrix} F(A) & A^2 & A & I \\ F(\lambda_1) & \lambda_1^2 & \lambda_1 & 1 \\ F'(\lambda_1) & 2\lambda_1 & 1 & 0 \\ F''(\lambda_1) & 2 & 0 & 0 \end{vmatrix} = 0 \quad \text{or}$$

$$F(A) = L_1^{(0)}(A)\,F(\lambda_1) + L_1^{(1)}(A)\,F'(\lambda_1) + L_1^{(2)}(A)\,F''(\lambda_1)$$

where $L_1^{(0)} = I$, $L_1^{(1)} = A - \lambda I$, $L_1^{(2)} = \frac{1}{2}(A-\lambda I)^2$

<u>Example</u> 4. Evaluate e^{At} where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

This matrix has a double eigenvalue -1 and a simple one 3. The reduced characteristic polynomial is found to be $(\lambda+1)^2(\lambda+3)$. With $F(\lambda_1) = e^{\lambda_1 t}$, $F'(\lambda_1) = te^{\lambda_1 t}$ we calculate

$$e^{At} = \left[I - \frac{(A+I)^2}{(-1-3)^2}\right]e^{-t} + \frac{(A+I)(A-3I)}{-1-3}\,te^{-t} + \frac{(A+I)^2}{(3+1)^2}$$

$$= \frac{1}{16}\begin{bmatrix} 8 & -4 & -6 \\ -4 & 14 & -3 \\ -8 & -4 & 10 \end{bmatrix} e^{-t} + \frac{1}{4}\begin{bmatrix} 0 & 4 & -2 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix} te^{-t} + \frac{1}{16}\begin{bmatrix} 8 & 4 & 6 \\ 4 & 2 & 3 \\ 8 & 4 & 6 \end{bmatrix} e^{3t}$$

4 Systems of homogeneous linear differential equations with constant coefficients

A system of n linear first order DE's of the form

$\dot{x}_1 = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n$

$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n$

— — — — — — — — — —

$\dot{x}_n = a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n$

with initial conditions $x_1(0) = c_1$, $x_2(0) = c_2$ etc can be written succinctly in matrix notation as

$$\dot{x} = Ax, \ x(0)=c$$

Formally introducing the matrix function

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^rt^r}{r!} + \cdots$$

it is easily verified that

$$\frac{d}{dt} e^{At} = Ae^{At} \quad , \quad e^{At}\Big|_{t=0} = I$$

The solution of $\dot{x} = Ax$ hence simply is

$$x = e^{At} c$$

In the preceeding section it has been shown how to evaluate e^{At}.

Examples

1. The system $\dot{x}_1 = 2x_1 - 3x_2 + 3x_3$

$\dot{x}_2 = 4x_1 - 5x_2 + 3x_3$

$\dot{x}_3 = 4x_1 - 4x_2 + 2x_3$ has the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} e^{t} \right\} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

where the expression for e^{At} was obtained in example 1 of section 3.

2\. The system $\dot{x}_1 = 2x_1 + x_2 - x_3$

$$\dot{x}_2 = x_1 + 2x_2 - x_3$$

$\dot{x}_3 = -x_1 - x_2 + 2x_3$ has the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{3}\left\{ \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} e^{t} + \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} e^{4t} \right\} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

The reader is reminded that a system containing derivatives of higher order can always be converted in a system of first order equations by the introduction of new variables.

Example Let

$$\ddot{x}_1 - x_1 + 5\dot{x}_2 = 0$$

$$\ddot{x}_2 - 4x_2 - 2\dot{x}_1 = 0$$

Introduce $x_3 = \dot{x}_1$ and $x_4 = \dot{x}_2$, then

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= x_1 - 5x_4 \\ \dot{x}_4 &= 4x_2 + 2x_3 \end{aligned}$$

5 Systems of non-homogeneous linear differential equations with constant coefficients

We want now to solve the matrix DE

$$\dot{x} = Ax + f(t)$$

The solution of the homogeneous equation is $e^{At}c$. To obtain a particular solution of the non-homogeneous equation put $x_p = e^{At}v(t)$. Hence $\dot{x}_p = Ae^{At}v + e^{At}\dot{v}$. Substituting

$$Ae^{At}v + e^{At}\dot{v} = Ae^{At}v + f(t)$$

$$e^{At}\dot{v} = f$$

and we would like to write

$$\dot{v} = e^{-At}f.$$

To show that this is correct let $F(A) = e^{-At}\cdot e^{At}$ and apply the Lagrange - Sylvester Theorem. We find if all eigenvalues are different in the reduced polynomial

$$F(A) = L_1(A)e^{-\lambda_1 t}\cdot e^{\lambda_1 t} + L_2(A)e^{-\lambda_2 t}\cdot e^{\lambda_2 t} + \cdots + L_s(A)e^{-\lambda_s t}\cdot e^{\lambda_s t}$$

$$= L_1(A) + L_2(A) + \cdots + L_s(A) = I.$$

The same result should hold true if we take a limit and let some eigenvalues coincide.

Integrating we have

$$v = \int e^{-At}f(t)dt \quad \text{and} \quad x_p = e^{At}\int e^{-At}f(t)dt$$

Again using the Lagrange - Sylvester Theorem we have for

$$F(A) = e^{At}\int e^{-At}f(t)dt = x_p$$

$$x_p = L_1(A)e^{\lambda_1 t}\int e^{-\lambda_1 t}f(t)dt + \cdots + L_s(A)e^{\lambda_s t}\int e^{-\lambda_s t}f(t)dt$$

if the reduced characteristic equation has s simple eigenvalues. In the case that multiple eigenvalues occur appropriate changes have to be made along the lines discussed in section 3.

<u>Example</u> $\dot{x} = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$

The eigenvalues are 1, 2 and the Lagrangian coefficients are found to be $L_1 = \begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}$ $L_2 = \begin{bmatrix} -3 & 3 \\ -4 & 4 \end{bmatrix}$

Furthermore $e^{\lambda_1 t}\int e^{-\lambda_1 t} f(t)dt = e^{t}\int e^{-3t}dt \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$

$$e^{\lambda_2 t}\int e^{-\lambda_2 t} f(t)dt = e^{2t}\int e^{-4t}dt \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

so that

$$x_p = \left\{ -\frac{1}{3}\begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} -3 & 3 \\ -4 & 4 \end{bmatrix} \right\} \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} = \frac{1}{4}\begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}$$

5 The equation $\ddot{x} + Ax = 0$

This equation of course can be solved by introducing the new variables $\dot{x} = y$ as mentioned in section 4. This however doubles the order of the matrix of coefficients. Since the matrix equation $\ddot{x} + Ax = 0$ has many important applications (coupled oscillations for instance) it merits a separate discussion. It will have become clear that the way to solve a matrix DE is to treat it as a scalar DE and then to give an interpretation of the result in terms of matrices. If A and x were scalars $\ddot{x} + Ax = 0$ with initial conditions $x(0) = c$, $\dot{x}(0) = \dot{c}$ would have the solution

$$x = \cos(t\sqrt{A})\, c + \frac{\sin(t\sqrt{A})}{\sqrt{A}}\, \dot{c}$$

In the interpretation of this result in terms of a matrix A a difficulty seems to arise. In the arguments of the cosine and sine $\sqrt{A}$ occurs and since $\sqrt{A}$ has no power series expansion the Lagrange-Sylvester theorem cannot be applied to it. Fortunately however if A is a scalar, we have

$$\cos(t\sqrt{A}) = 1 - \frac{At^2}{2!} + \frac{A^2t^4}{4!} - \frac{A^3t^6}{6!} + \cdots$$

$$\frac{\sin(t\sqrt{A})}{\sqrt{A}} = t - \frac{At^3}{3!} + \frac{A^2t^5}{5!} - \frac{A^3t^7}{7!} + \cdots$$

and we can use these series, replacing the first terms by I and It respectively to define $\cos(t\sqrt{A})$ and $\frac{\sin(t\sqrt{A})}{\sqrt{A}}$ if A is a matrix. It is easily verified by substitution that these functions both satisfy the equation $\ddot{x} + Ax = 0$. We now invoke the Lagrange-Sylvester theorem to write them as polynomials in A. If all roots of the reduced characteristic equation are simple one obtains the solution in the form

$$X = \left(\sum L_k(A)\cos\omega_k t\right)c + \left(\sum L_k(A)\frac{\sin\omega_k t}{\omega_k}\right)\dot{c}$$

where $\omega_k = \sqrt{\lambda_k}$

<u>Example</u> $\ddot{X}_1 = 2X_1 - 3X_2$ the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$

$\ddot{X}_2 = X_1 - 2X_2$

The Lagrangian coefficients are

$$L_1(A) = \frac{1}{2}(A + I) = \frac{1}{2}\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}, \quad L_2 = -\frac{1}{2}(A - I) = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}$$

Hence

$$X = \left\{\frac{1}{2}\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}(\cos t + \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}\cos it\right\}c + \left\{\frac{1}{2}\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\sin t + \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}\frac{\sin it}{i}\right\}\dot{c}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\cos t + \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}\cosh t\right\}c + \frac{1}{2}\left\{\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\sin t + \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}\sinh t\right\}\dot{c}$$

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LINEAR ALGEBRA

Final Examination

Name BOB MARKS Box 385

Work all problems.

Do not write below this line

(10)	1	10	(10)	7	7
(10)	2	10	(10)	8	5
(10)	3	10	(10)	9	4
(10)	4	7	(10)	10	8
(10)	5	6	(100)	Total	77
(10)	6	10		GRADE	C+

1. Find the complete solution of

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 + 3x_5 &= 1 \\ 2x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 &= 2 \\ 3x_1 + 2x_2 - 4x_3 + 3x_4 - 9x_5 &= 3 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 2 & -1 & 2 & 2 & 6 & 2 \\ 3 & 2 & -4 & 3 & -9 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & -3 & 6 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -18 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & -2 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -18 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -18 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$\therefore$ $x_5 = 0$

$x_2 - 2x_3 = 0 \Rightarrow x_2 = 2x_3$

$x_1 + x_4 + 3x_5 = 1 \Rightarrow x_1 = -x_4 - 3x_5 + 1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \overset{\vec{0}}{\cancel{\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_5}} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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2. Invert the matrix

$$A = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & +1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

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3. Show that

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & 2a & 3a^2 \\ 1 & b & b^2 & b^3 \\ 0 & 1 & 2b & 3b^2 \end{vmatrix} = (b - a)^{\cancel{3}4} = b^4 - 4b^3a + 6b^2a^2 - 4ba^3 + a^4$$

at least expand with respect to first <u>column</u>

$$= \begin{vmatrix} 1 & 2a & 3a^2 \\ b & b^2 & b^3 \\ 1 & 2b & 3b^2 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ 1 & 2a & 3a^2 \\ 1 & 2b & 3b^2 \end{vmatrix}$$

$$= b\begin{vmatrix} 1 & 2a & 3a^2 \\ 1 & b & b^2 \\ 1 & 2b & 3b^2 \end{vmatrix} + a\begin{vmatrix} 1 & a & a^2 \\ 1 & 2a & 3a^2 \\ 1 & 2b & 3b^2 \end{vmatrix}$$

$$= b\left[\begin{vmatrix} b & b^2 \\ 2b & 3b^2 \end{vmatrix} - \begin{vmatrix} 2a & 3a^2 \\ 2b & 3b^2 \end{vmatrix} + \begin{vmatrix} 2a & 3a^2 \\ b & b^2 \end{vmatrix}\right]$$

$$+ a\left[\begin{vmatrix} 2a & 3a^2 \\ 2b & 3b^2 \end{vmatrix} - \begin{vmatrix} a & a^2 \\ 2b & 3b^2 \end{vmatrix} + \begin{vmatrix} a & a^2 \\ 2a & 3a^2 \end{vmatrix}\right]$$

$$= b\left[b^2\begin{vmatrix} 1 & b \\ 2 & 3b \end{vmatrix} - ab\begin{vmatrix} 2 & 3a \\ 2 & 3b \end{vmatrix} + ab\begin{vmatrix} 2 & 3a \\ 1 & b \end{vmatrix}\right]$$

$$+ a\left[ab\begin{vmatrix} 2 & 3a \\ 2 & 3b \end{vmatrix} - ab\begin{vmatrix} 1 & a \\ 2 & 3b \end{vmatrix} + a^2\begin{vmatrix} 1 & a \\ 2 & 3a \end{vmatrix}\right]$$

$$= b^3\begin{vmatrix} 1 & b \\ 2 & 3b \end{vmatrix} - ab^2\begin{vmatrix} 2 & 3a \\ 2 & 3b \end{vmatrix} + ab^2\begin{vmatrix} 2 & 3a \\ 1 & b \end{vmatrix} + a^2b\begin{vmatrix} 2 & 3a \\ 2 & 3b \end{vmatrix} - a^2b\begin{vmatrix} 1 & a \\ 2 & 3b \end{vmatrix} + a^3\begin{vmatrix} 1 & a \\ 2 & 3a \end{vmatrix}$$

$$= b^4\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 6ab^2\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} + ab^2\begin{vmatrix} 2 & 3a \\ 1 & b \end{vmatrix} + 6a^2b\begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} - a^2b\begin{vmatrix} 1 & a \\ 2 & 3b \end{vmatrix} + a^4\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= b^4 - 6ab^2(b-a) + ab^2(2b-3a) + 6a^2b(b-a) - a^2b(3b-2a) + a^4$$

$$= b^4 - 6ab^3 + \cancel{6a^2b^2} + 2ab^3 - 3a^2b^2 + \cancel{6a^2b^2} - 6a^3b + 2a^3b + a^4$$

$$= b^4 - 4b^3a + 6b^2a^2 - 4ba^3 + a^4 = (b-a)^4$$

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4.

Given the vectors $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 6 \\ -5 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 1 \end{bmatrix}$, $\vec{c} = \begin{bmatrix} 2 \\ -3 \\ 9 \\ -1 \end{bmatrix}$, $\vec{d} = \begin{bmatrix} 2 \\ -5 \\ 7 \\ 5 \end{bmatrix}$

(i) Determine whether they are linearly dependent.

(ii) Find a basis for the subspace spanned by $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -2 & -3 & -5 \\ 6 & 4 & 9 & 7 \\ -5 & 1 & -1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & -2 & -3 & -5 \\ -5 & 1 & -1 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ -5 & 1 & -5 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \therefore \vec{a}, \vec{b}, \vec{c}\, \vec{d}$$ ARE LINEARLY DEPENDENT

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & 6 & 5 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow rr = 3$$

TRY $\vec{a}\vec{b}\vec{c}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -3 \\ 6 & 4 & 9 \\ -5 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ FROM

$\Rightarrow a, b, c$ ARE LIN. INDEP.

$\therefore$ LET BASIS $= \vec{a}, \vec{b}, \vec{c}$

$$\vec{a} = 2\vec{c} - 3\vec{b}$$

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5. a. Let $a = (a_1, a_2, a_3)$ be a vector in $\mathcal{R}_3$. Determine whether the following are subspaces. If not, give one reason why.

✓ (i) The set of all vectors for which $a_1 = a_2 = a_3$.

IS A SUBSPACE

✓ (ii) The set of all vectors for which $a_1 = 0$

IS A SUBSPACE

✓ (iii) The set of all vectors for which $a_1^2 = a_2^2$.

NOT CLOSED UNDER ADDITION

$$EX \Rightarrow \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} ; \quad 4^2 \neq 0^2$$

b. Let $\mathcal{R}_{nn}$ be the vector space of all matrices over $\mathcal{R}$. Is the set of all matrices A that commute with a given matrix B : AB = BA a subspace? If not give one reason why.

$R_{nn} \in R ; A, B \in R_{nn}$

$\cancel{IB = BI}$

$E_{in} B = B E_{in}$

$(I + E_{in}) B = B (I + E_{in})$

$$EX \; \overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{I+E_{12}}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} \neq \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$$

$\Rightarrow$ ~~NO, NOT~~ IS CLOSED UNDER ADDITION

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 2 & 0 & \cdots & 0 \\ & & 3 & & \\ 1 & 0 & 0 & \cdots & 2 \end{bmatrix}$$

c. Is the set of all trigonometric polynomials of the form

$$\frac{1}{2} a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

a vector space? If not give one reason why.

$\exists$ NO ADDITIVE IDENTITY FOR ANY GIVEN $\vec{X}$ (NO $\vec{0}$) ??? Take $a_n = b_n = a_0 = 0$

$$(\alpha A_1 + \beta A_2) B = \alpha A_1 B + \beta A_2 B = \alpha B A_1 + \beta B A_2 = B(\alpha A_1 + \beta A_2)$$

6. If A and B are non-singular square matrices of the same order and p a scalar, indicate which of the following identities or conclusions are wrong and make appropriate corrections

WRONG $(A + B)(A - B) = A^2 - B^2$

$$(A+B)(A-B)=(A+B)A-(A+B)B$$
$$=A^2+BA-AB-B^2$$ ✓

WRONG $AX = B \Rightarrow X = BA^{-1}$

$$AX=B$$
$$A^{-1}AX=A^{-1}B$$
$$X=A^{-1}B$$ ✓

WRONG $\det (pA) = p \det A$

GIVEN A IS OF ORDER nn

$$\det(pA)=p^n \det A$$ ✓

OK $[(AB)^{-1}]^T = (A^{-1})^T (B^{-1})^T$

$$(AB)^{-1}=B^{-1}A^{-1}$$
$$[(AB)^{-1}]^T=(A^{-1})^T(B^{-1})^T$$ ✓

WRONG $A \operatorname{adj} A = \det A$

$$A \operatorname{adj} A = I \det A$$ ✓

7. Given the vectors

$$\vec{a} = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}2\\3\\0\\1\end{bmatrix}, \quad \vec{c} = \begin{bmatrix}0\\1\\-1\\0\end{bmatrix}, \quad \vec{d} = \begin{bmatrix}3\\1\\2\\1\end{bmatrix}, \quad \vec{e} = \begin{bmatrix}1\\-3\\1\\2\end{bmatrix}, \quad \vec{f} = \begin{bmatrix}2\\-1\\2\\1\end{bmatrix}$$

The triple $\vec{a}, \vec{b}, \vec{c}$ is a basis for the subspace S and the triple $\vec{d}, \vec{e}, \vec{f}$ is a basis for the subspace T. Determine the dimension of the intersection $S \cap T$ of S and T and find a basis for it.

$$\begin{bmatrix}1&2&0&3&1&2\\1&3&1&1&-3&-1\\1&0&-1&2&1&2\\0&1&0&1&2&1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\-\beta_1\\-\beta_2\\-\beta_3\end{bmatrix} = \vec{0}$$

$$\Rightarrow\begin{bmatrix}1&2&0&3&1&2\\0&1&1&-2&-4&-3\\0&2&1&1&0&0\\0&0&-1&1&4&2\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&-2&7&9&8\\0&1&1&-2&-4&-3\\0&0&+1&-5&-8&-6\\0&0&-1&1&4&3\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0&-3&-7&-4\\0&1&0&-1&0&0\\0&0&1&-5&-8&-6\\0&0&0&-4&-4&-3\end{bmatrix}$$

$\therefore\ \alpha_1+3\beta_1+7\beta_2+4\beta_3=0$

$\rightarrow \underline{\alpha_1=-3\beta_1-7\beta_2-4\beta_3}$

$\alpha_2+\beta_1=0 \Rightarrow \underline{\alpha_2=-\beta_1}$

$\alpha_3=-5\beta_1-8\beta_2-6\beta_3$

$\bullet\ 4\beta_1+4\beta_2+3\beta_3=0$

$\vec{w}=\alpha_1 a+\alpha_2 b+\alpha_3 c$

$=-(3\beta_1+7\beta_2+4\beta_3)a-\beta_1 b-(5\beta_1+8\beta_2+6\beta_3)\vec{c}$

$=\beta_1(-3a-b-5c)+\beta_2(-7a-8c)+\beta_3(-4a-6c)$

~~3RD DIMENSION~~

throw out β_3 or β_1

BASIS VECTORS:

$\vec{V}_1=-3\vec{a}-\vec{b}-5\vec{c}$

$\vec{V}_2=-7\vec{a}-8\vec{c}$

$\vec{V}_3=-4\vec{a}-6\vec{c}$

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8. Find all the eigenvalues and eigenvectors of the matrix.

$$A = \begin{bmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{bmatrix}$$

Determine its characteristic equation and its minimum equation.

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ -1 & 3-\lambda & -1 \\ 2 & -4 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2 + 4 + 4 - 4(2-\lambda) - 2(3-\lambda) - 2(3-\lambda)$$

$$= (2-\lambda)(9-6\lambda+\lambda^2) + \cancel{8-8} + 4\lambda \cancel{-6} + 2\lambda \cancel{-6} + 2\lambda$$

$$= 18 \cancel{-12\lambda} + 2\lambda^2 - 9\lambda \cancel{+6\lambda^2} - \lambda^3 - 12 + 8\lambda = 0$$

$$= +\lambda^3 - 8\lambda^2 + 13\lambda + 6 = 0$$

$$= (\lambda-6)(\lambda-1)(\lambda-1) = 0 \leftarrow \text{CHAR EQ}$$

∴ EIGEN VALUES = 6, 1, 1 ✓

$$\Rightarrow (A-6I)(A-I)^2 = 0$$

$$(A-6I)(A-I) \overset{?}{=} 0$$

$$\begin{bmatrix} -4 & -2 & 1 \\ -1 & -3 & -1 \\ 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 2 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ $(\lambda-6)(\lambda-1)=0$ IS MIN. EQ ✓

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9. (i) Show that two similar matrices have the same characteristic equation.

(ii) The matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has eigenvalues $-1, 5$.

Determine a similarity transformation P such that $P^{-1}AP$ is a diagonal matrix D. Invert P, evaluate $P^{-1}AP$ and verify that it is diagonal and has the eigenvalues of A on the main diagonal.

~~i) $P_1^{-1}A_1P_1 = D_1 \ni P_1$ = MATRIX OF EIGEN VECTORS AND d_{nn} = EIGEN VALUES OF A~~

~~$P_2^{-1}A_2P_2 = D_2$. $D_1 = D_2 \Rightarrow$ A & B ARE SIMILAR, AND $\therefore$ HAVE SAME EIGEN VALUES~~

ii) $\lambda = -1 \Rightarrow \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \therefore E_1 = \alpha_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 5 \Rightarrow \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \therefore E_2 = \alpha_2 \cancel{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$

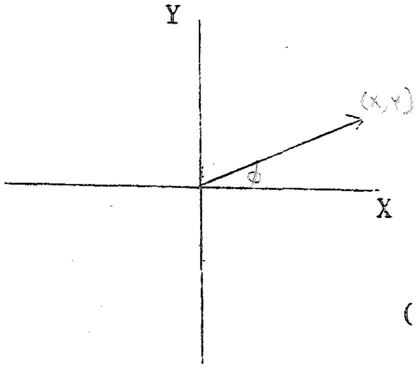
$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ +1 & +1 & 0 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow P^{-1} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -9 & -8 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 10 \\ 0 & 5 \end{bmatrix}$

4.

10.



The vector $\begin{bmatrix} x \\ y \end{bmatrix}$ is subjected to the following transformations:

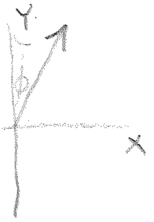
(i) Reflection in the X axis : T_1

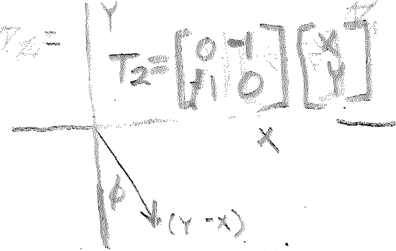
(ii) Rotation over 90^o : T_2

(iii) Reflection in the line x = y : T_3

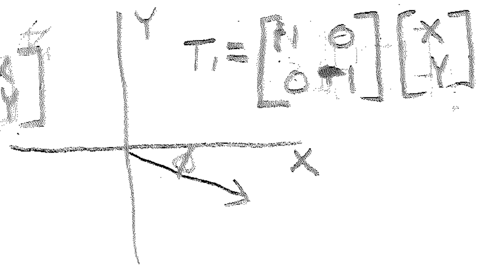
Find in a direct way the matrices associated with these transformations. Show that $T_3 = T_2 T_1$ and give a geometric interpretation of $T_4 = T_1 T_2$.

$T_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$





$T_2 = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$



$T_1 = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$

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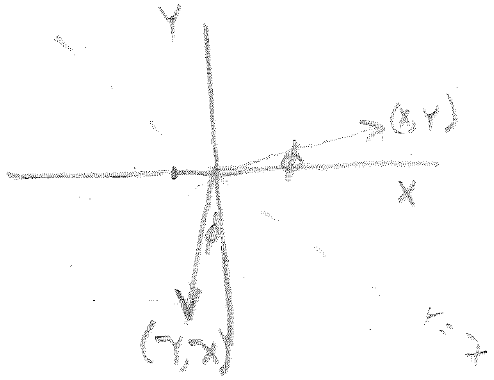
T_2

T_2T_1

$= T_3$

$$T_2T_1 \Rightarrow \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & +1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = T_3$$

$$T_1T_2 \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} -Y \\ -X \end{bmatrix}$$



REFECTION IN LINE $Y = -X$

Marks 385

Eqn : $(2-\lambda)^2(5-\lambda)=0$ or $\lambda^3-9\lambda^2+24\lambda-20=0$ $\lambda_1=\lambda_2=2$, $\lambda_3=5$

$\begin{bmatrix}0&1&2\\0&0&3\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=0 \leftarrow$ eigenvector v_1 $\lambda_3=5$ $\begin{bmatrix}-3&1&2\\0&-3&3\\0&0&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=0 \leftarrow$ eigenvector v_3

$\therefore A^2-9A+24I$

$$\frac{1}{20}\begin{bmatrix}2&1&2\\0&2&3\\0&0&5\end{bmatrix}^2-\frac{9}{20}\begin{bmatrix}2&1&2\\0&2&3\\0&0&5\end{bmatrix}+\frac{24}{20}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\frac{1}{20}\begin{bmatrix}4&4&17\\0&4&21\\0&0&25\end{bmatrix}-\frac{9}{20}A+\frac{24}{20}I=\frac{1}{20}\begin{bmatrix}10&-5\\0&10\\0&0\end{bmatrix}$$

$$\begin{bmatrix}1-\lambda&1&-2\\1&1-\lambda&-2\\-2&-2&4-\lambda\end{bmatrix}=\begin{aligned}&(1-\lambda)(\lambda^2-5\lambda+4-4)-1(4-\lambda-4)-2(-2+2-2\lambda)=\\&(1-\lambda)\lambda(\lambda-5)+\lambda+4\lambda=\lambda(-\lambda^2+6\lambda-5+5)=-\lambda^2(\lambda-6)\end{aligned}$$

$\lambda_1=\lambda_2=0$ $\lambda_3=6$

$\lambda=0$ $\begin{bmatrix}1&1&-2\\1&1&-2\\-2&-2&4\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=0$, $\begin{bmatrix}1&1&-2\\1&1&-2\\-2&-2&4\end{bmatrix}\begin{bmatrix}1\\-1\\0\end{bmatrix}=0$ $\lambda=6$ $\begin{bmatrix}-5&1&-2\\1&-5&-2\\-2&-2&-2\end{bmatrix}\begin{bmatrix}1\\1\\-2\end{bmatrix}=0$

eigenvectors $\uparrow v_1$ $\uparrow v_2$ $\uparrow v_3$

For min eqn try $A(A-6I)=\begin{bmatrix}1&1&-2\\1&1&-2\\-2&-2&4\end{bmatrix}\begin{bmatrix}-5&1&-2\\1&-5&-2\\-2&-2&-2\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

[illegible]

1) Find v_1 v_2 v_3

$$\underline{\lambda=-2}\begin{bmatrix}2&-2&2\\-3&3&3\\-1&1&5\end{bmatrix}\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\uparrow v_1}=0 \qquad \underline{\lambda=2}\begin{bmatrix}-2&-2&2\\-3&-1&3\\-1&1&1\end{bmatrix}\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{v_2\uparrow}=0 \qquad \underline{\lambda_3=4}\begin{bmatrix}-4&-2&2\\-3&-3&3\\-1&1&-1\end{bmatrix}\underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{v_3\uparrow}=0$$

$$P=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$$

for P^{-1}:

$$\begin{bmatrix}1&1&0&1&0&0\\1&0&1&0&1&0\\0&1&1&0&0&1\end{bmatrix}\overset{R}{\sim}\begin{bmatrix}1&1&0&1&0&0\\0&-1&1&-1&1&0\\0&1&1&0&0&1\end{bmatrix}\overset{R}{\sim}\begin{bmatrix}1&0&1&0&1&0\\0&-1&1&-1&1&0\\0&0&2&-1&1&1\end{bmatrix}\overset{R}{\sim}\begin{bmatrix}1&0&1&0&1&0\\0&1&-1&1&-1&0\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix}$$

$$\overset{R}{\sim}\begin{bmatrix}1&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&1&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix} \qquad P^{-1}=\frac{1}{2}\begin{bmatrix}1&1&-1\\1&-1&1\\-1&1&1\end{bmatrix}$$

$$P^{-1}AP=\frac{1}{2}\begin{bmatrix}1&1&-1\\1&-1&1\\-1&1&1\end{bmatrix}\begin{bmatrix}0&-2&2\\-3&1&3\\-1&1&3\end{bmatrix}P=\frac{1}{2}\begin{bmatrix}-2&-2&2\\2&-2&2\\-4&4&4\end{bmatrix}\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}=\begin{bmatrix}-2&0&0\\0&2&0\\0&0&4\end{bmatrix}$$

This quiz did not have a problem on linear transformations. That does <u>not</u> mean that no such problem will be on the final. On the contrary!